Notes in Economic Growth and Fluctuations

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1 Introduction

The purpose of these notes is to provide a comprehensive introduction to modern macroeconomics for Master's students and researchers who may not be familiar with the field. Modern macroeconomics is characterized by the use of micro-founded behaviors of economic agents, such as households and firms, within dynamic environments. This analytical framework represents a significant departure from traditional models like IS-LM or AS-AD, which are typically studied at the undergraduate level. Given the substantial gap between undergraduate macroeconomics and more advanced, contemporary approaches, I deemed it necessary to prepare these notes. Additionally, this work is intended to facilitate my teaching responsibilities in the classroom.

These notes are organized around four main themes, which forms the building blocks of this document:

1. A quick guide for methods in deterministic dynamics (Part I: Methods in Dynamic Economics), covering tools like ordinary differentials equations, difference equations and dynamic optimization
2. An overview of the benchmark model and their applications in growth theory (Part II: Benchmark models) presenting key models in dynamic macroeconomics. The Solow model introduces fundamental concepts in macrodynamics, while the Ramsey-Cass-Koopmans model (often referred to as the Ramsey model) and the Overlapping Generations model (OLG) are among the most widely utilized models.
3. some notes on Endogenous Growth Theory (Part III: Endogenous Growth Theory) summarizing major contributions in this area, ranging from the AK model to models of technological change.
4. Topics in Aggregate Fluctuations (to be written).

Readers should bear in mind that a model is primarily a tool for understanding real-world problems. Just as a tool can vary in its sharpness, a model can be more or less suited to studying a particular issue. Theoretical developments aim to refine these tools, while empirical methods assess their effectiveness in explaining different types of problems. In these notes, my goal is to provide students with a solid foundation for theoretical work in economics. The applications are potentially infinite (a major conclusion of Endogenous Growth Theory). Although additional applications may be incorporated over time, I believe this volume is already sufficiently comprehensive and beneficial to students in its current form. This is an initial version, and it is likely to evolve, incorporating new contents (topics, code blocks ...) or revised structures within chapters.

Many researchers and educators have contributed to the extensive literature on the topics covered here. While I have endeavored to write as accurately and originally as possible, I acknowledge that some formulations or sections may overlap with existing works. I apologize in advance for any such overlaps and will continue to refine this document until it reaches the desired standard. All errors are, of course, my own.

Part I

Methods

2 Ordinary Differentials Equations

In this chapter, we review solution methods of linear differential equations and systems of linear differential equations, also called Ordinary Differential Equations (ODE). These are essential when studying continuous time problem in macroeconomic dynamics. The chapter is organized as follows: we will first review some basic definitions and then proceed to examine solutions methods of first-order and second-order differential equations. We finally discuss the generalizations of the solution methods to a system of n first-order differential equations.

2.1 Definitions

A *differential equation* is a mathematical equation derivend from an unknown function of one or more variables, which connects the function itself and its derivatives of various degrees.

The solution of a simple equation (or a system of equations) is a constant or a set of constants that satisfy these equations. In contrast, the solution of a differential equation (or a system of differential equations) is a function, or a set of functions that, along with their derivatives, satisfy the differential equation or the system of differential equations. The equations we shal consider are all functions of time t, which is assumed to be a continuous variable defined on the set of real numbers $\mathbb{R}$.

Consider for a example the solution to the differential equation:[1]

$$\dot{y}(t) = \frac{dy}{dt} = a$$

is a function $y(t)$, the first derivative of which with respect to time is equal to a. The general solution to this differential equation is the function

$$y(t) = at + c$$

where c is an arbitrary constant. A particular solution can then be found using a boundary condition, such as an **initial condition** known at time $t = 0$, y_0 taken as given.

To find the general solution, rewrite the initial equation as :

[1] We use the conventional notation where t enters as an argument for each variable that depends to time and the "dot" over a variable means the time derivative of the considered variable.

$$\frac{\dot{y}(t)}{y(t)} = a \Rightarrow \frac{1}{y(t)}\frac{dy(t)}{dt} = a \Rightarrow \frac{dy}{y} = adt$$

Which can be integrated:

$$\int \frac{dy}{y} = \int adt + c \Rightarrow ln(y(t)) = at \Rightarrow y(t) = c\exp^{at}$$

with c an arbitrary constant.To determine this constant and find the particular solution, use the information $y(0) = y_0$ given:

$$y(0) = y_0 = c\exp^{a\cdot 0} \Rightarrow c = y_0$$

The final solution has then the form:

$$y(t) = y_0 exp^{at}$$

Note that we can differentiate with respect to t the RHS of the solution : $\dot{y}(t) = ay_0 exp^{at} \equiv ay(t)$. We have indeed found the solution.

We could also look at a differential equation of the form $\ddot{y}(t) = \frac{d^2y(t)}{dt^2} = a$ which is a second-order differential equation and so on ...

A solution of a differential equation is a function $y(t)$, which along with its derivatives, satisfies the differential equation. A *particular* solution requires the determination of the arbitrary constant or a constants of integration.

Differential equations are classified by their porder, which is none other than the order of the highest derivative that appears in the equation.

A differential equation is *linear* if the unknown function $y(t)$ and its derivatives are linear. Otherwise, it is nonlinear.

A differential equation can be solved by a method known as *separation of variables* if it can be written as a term that contains a function of only y, equated to a term that contains a function only of t. For example, $g(y(t))\dot{y}(t) = f(t)$ can be written as:

$$g(y)dy = f(t)dt$$

The solution can be found using the integration techniques used above.

2.2 First-Order Linear Differential Equations

We may found two types of linear differential equations: those with constant coefficients or those with variable coefficients.

2.2.1 Constant Coefficients

This category of linear ODE takes the form:

$$\dot{y}(t) + ay(t) = b \tag{2.1}$$

with a and b some constants. To find the function that satisfies (2.1), note that

$$\frac{d\left(e^{at}y(t)\right)}{dt} = ae^{at}y(t) + e^{at}\dot{y}(t) = e^{at}[\dot{y}(t) + ay(t)] \tag{2.2}$$

If the differential equation (2.1) is multiplied by e^{at}, its left-hand side is an exact differential equation) (i.e., the total differential of a function with respect to t). The function e^{at} is called the integrating factor. Multiplying both sides of (2.2) by dt, we get

$$d\left(e^{at}y(t)\right) = be^{at}dt$$

whose integral is

$$e^{at}y(t) = e^{at}\frac{b}{a} + c$$

,

where c is the constant of integration. Multiplying both sides of this expression by e^{-at}, we get

$$y(t) = \frac{b}{a} + ce^{-at} \tag{2.3}$$

as the family of functions that satisfy the differential equation (2.1). This family is called the *general solution* of (2.1).

To determine the constant of integration c, we need to know the value of the function at some point in time. For example, if we know that at time $t = 0$, $y(0) = y_0$, then

$$y_0 = \frac{b}{a} + c$$

which implies

$$c = y_0 - \frac{b}{a}$$

The general solution of the differential equation (2.1) that satisfies $y(0) = y_0$ is then given by

$$y(t) = y_0 e^{-at} + \left(1 - e^{-at}\right)\frac{b}{a} = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{-at}$$

In conclusion, to solve a linear first-order differential equation with constant coefficients, multiply it by the integrating factor and integrate it. To calculate the constant of integration, use the value of the function at some point. The point that is used is called an *initial condition* or a *terminal condition*, or more generally, a *boundary condition*.

2.2.2 Variable Right-Hand Side

If the right-hand side of (2.1) is not constant but a known function of time, the solution method is similar. We multiply by the integrating factor and take the integral.

For example, for the differential equation

$$\dot{y}(t) + ay(t) = be^{\lambda t} \tag{2.4}$$

multiplying by the integrating factor and separating the variables results in

$$d\left(e^{at} y\right) = be^{(a+\lambda)t} dt \tag{2.5}$$

Taking the integral of both sides of (2.5) yields

$$e^{at} y(t) = \frac{b}{a+\lambda} e^{(a+\lambda)t} + c$$

Dividing both sides by the integrating factor, we get the solution

$$y(t) = \frac{b}{a+\lambda} e^{\lambda t} + ce^{-at} \tag{2.6}$$

Equation (2.6) is the family of functions satisfying (2.4). The unknown constant c can again be determined by a boundary condition.

2.2.3 Variable Coefficients

The general form of a first-order linear differential equation is

$$\dot{y}(t) + a(t)y(t) = b(t) \tag{2.7}$$

where $a(t)$ and $b(t)$ are known functions, and we seek the function $y(t)$. The function $b(t)$ is often called a forcing term and is considered exogenous. The integrating factor in this case is $e^{\int a(t)dt}$, because

$$\frac{d\left(y(t)e^{\int a(t)dt}\right)}{dt} = e^{\int a(t)dt}[\dot{y}(t) + a(t)y(t)]$$

Thus, multiplying (2.7) by this integrating factor and taking the integral, we get

$$y(t)e^{\int a(t)dt} = \int b(t)e^{\int a(t)dt}dt + c$$

Dividing (C.21) by the integrating factor, we finally get

$$y(t) = e^{-\int a(t)dt} \int b(t)e^{\int a(t)dt}dt + e^{-\int a(t)dt}c \tag{2.8}$$

where c is the constant of integration. (2.8) is the general solution of (2.7). A particular solution requires a boundary condition that will determine the unknown constant c.

Note that it is not advisable to apply the solution (2.8) to any equation. It is simpler in many cases to multiply by the integrating factor and take the integral.

2.2.4 Homogeneous and Nonhomogeneous Differential Equations

If $b = 0$ in (2.1), the differential equation to be solved is called homogeneous. Otherwise, it is called nonhomogeneous.

The general solution of a differential equation consists of the sum of the general solution to the relevant homogeneous differential equation (i.e., setting $b = 0$ and solving, and then adding to this solution a particular solution of the general equation (2.1).

For example, the homogeneous equation derived from (2.1) is

$$\dot{y}(t) + ay(t) = 0 \tag{2.9}$$

The general solution of (2.9) is

$$y(t) = ce^{-at} \tag{2.10}$$

A particular solution (setting, for example, $\dot{y}(t) = 0$)) is

$$\bar{y} = \frac{b}{a} \tag{2.11}$$

Consequently, the general solution of the nonhomogeneous differential equation (2.1) is the sum of (2.10) and (2.11), that is, the general solution of the relevant homogeneous differential equation (sometimes called the complementary function) plus the particular solution for a constant y (otherwise known as the particular integral). The general solution is thus given by

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{-at}$$

This methodology is not generally necessary for solving first-order linear differential equations, but it becomes very useful for differential equations of order higher than one or for systems of first-order linear differential equations.

2.2.5 Convergence and Stability of First-Order Differential Equations

In many economic applications, we are interested in the behavior of the solution of a differential equation as the independent variable, usually time, tends to infinity. The value to which the solution converges is referred to as a stationary state, or steady state, or equilibrium state.

For example, from (2.3), which is the general solution of (2.1), for $a > 0$, we get

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\frac{b}{a} + ce^{-at}\right) = \frac{b}{a}.$$

The particular integral of the differential equation (2.1) can therefore be interpreted economically as the equilibrium state (or the steady state), which is the state toward which the variable y converges as time goes to infinity. This equilibrium is called a stable node. It is a stable equilibrium if y is a predetermined variable and only changes gradually, as postulated by the law of motion (C.26).

Assume now that $a < 0$. In this case, if the boundary condition y_0 is different from the steady value $\bar{y}$, then $y(t)$ (as determined by (C.26)) moves to plus or minus infinity, farther and farther away from the steady state. The only case in which this does not happen is when the boundary condition y_0 is equal to the steady state value $\bar{y} = b/a$. Then y remains constant at $\bar{y}$. However, this is an unstable equilibrium, called a saddle point. There is only one adjustment path that leads to it and this is for y to jump immediately to the steady state. If y is a non-predetermined variable (for example, a financial variable or any variable that can change abruptly and not gradually), then it can jump immediately to its steady state value.

2.3 Second-Order Linear Differential Equations

A second-order linear differential equation has the form

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) = h(t) \tag{2.12}$$

where $a(t), b(t)$, and $h(t)$ are known functions, and what is sought is the function $y(t)$. The forcing term in this case is the function $h(t)$. Equation (2.12) is referred to as the complete equation and is nonhomogenous. Related to (2.12) is a homogeneous differential equation in which $h(t) = 0$:

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) = 0 \tag{2.13}$$

which is called the reduced equation. The complete equation is nonhomogenous, whereas the reduced equation is homogeneous. The reduced equation is of interest because of the following two theorems.

Theorem 1

The general solution of the complete equation (2.12) is the sum of any particular solution of the complete equation and the general solution of the reduced equation (2.13).

Theorem 2

Any solution $y(t)$ of the reduced equation (2.13) on $t_0 \leq t \leq t_1$ can be expressed as a linear combination, $y(t) = c_1 y_1(t) + c_2 y_2(t), t_0 \leq t \leq t_1$, of any two particular solutions y_1, y_2 that are linearly independent.

2.3.1 Homogeneous Equations with Constant Coefficients

We first examine the differential equation (2.12), with constant coefficients, that is, $a(t) = a, b(t) = b$. Assume that $h(t) = 0$. The differential equation then takes the form

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = 0 \tag{2.14}$$

Inspired by the general solution of the first-order linear differential equation with constant coefficients, let us try the general solution

$$y(t) = ce^{rt}$$

with unknown constants c and r. This solution method is called the method of *undetermined coefficients*. This solution implies that

$$\dot{y}(t) = rce^{rt}, \ddot{y}(t) = r^2ce^{rt}$$

Substituting these expressions in (2.14), we get

$$ce^{rt}\left(r^2 + ar + b\right) = 0 \tag{2.15}$$

For a nonzero c, our trial solution satisfies (2.15) only if r is a root of the quadratic equation

$$r^2 + ar + b = 0 \tag{2.16}$$

Equation (2.16) is called the *characteristicequation* of (2.14). It has two roots:

$r_1, r_2 = \frac{-a\pm\sqrt{a^2-4b}}{2}$.

We distinguish three cases, depending on the value of the discriminant $a^2 - 4b$ of the characteristic equation (2.16).

- Case 1 $ a^{2}>4 b $ The discriminant is positive, and the roots are real and distinct. The general solution of (2.14) takes the form

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

where r_1, r_2 are the roots of the characteristic equation (2.16), and c_1, c_2 are arbitrary constants.

- Case 2 $ a^{2}<4 b$ The discriminant is negative, and the roots are a pair of complex conjugates:

$$r_1, r_2 = -\frac{a}{2} \pm i\frac{\sqrt{4b - a^2}}{2} = \alpha \pm i\beta$$

where $\alpha = -\frac{a}{2}$, and $\beta = \frac{\sqrt{4b-a^2}}{2}$. The general solution in this case is

$$y(t) = e^{\alpha t}\left(k_1 \cos \beta t + k_2 \sin \beta t\right)$$

where k_1, k_2 are arbitrary constants.

- Case 3 $a^2 = 4b$ The discriminant is equal to zero, and the two roots are the same and equal to $-a/2$. One can show that the general solution of (2.14) in this case takes the form

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}\left(c_1 + c_2 t\right)$$

where $r = -a/2$ is the double root of the characteristic equation (2.16), and c_1, c_2 are arbitrary constants.

2.3.2 Nonhomogeneous Equations with Constant Coefficients

We have already derived the solution of any homogeneous second-order linear differential equation with constant coefficients. To find the solution of a nonhomogeneous equation, we need a particular solution of the complete equation. If the complete equation is of the form

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = h$$

then a particular solution is the constant function

$$\bar{y} = \frac{h}{b}$$

The full solution is thus the sum of the general solution to the homogeneous equation plus the particular solution to the complete equation.

For differential equations with variable coefficients, more advanced methods, such as the method of variation of parameters, can be utilized for their solution.

2.4 A Pair of First-Order Linear Differential Equations

We next examine a case with extensive applications in macroeconomics: a pair of first-order linear differential equations of the form

$$\begin{aligned} \dot{x}(t) &= a_1 x(t) + b_1 y(t) + p(t) \\ \dot{y}(t) &= a_2 x(t) + b_2 y(t) + g(t) \end{aligned} \tag{2.17}$$

where a_1, a_2, b_1, b_2 are given constants, and $p(t), g(t)$ are given functions. The solution of the system of differential equations (2.17) will be two functions $x(t)$ and $y(t)$ that satisfy both differential equations.

The homogeneous system that corresponds to (2.17) is given by

$$\begin{aligned} \dot{x}(t) &= a_1 x(t) + b_1 y(t) \\ \dot{y}(t) &= a_2 x(t) + b_2 y(t) \end{aligned} \tag{2.18}$$

2.4.1 The Method of Substitution

One solution method is the method of substitution. Substituting $y(t)$ and its derivatives in the equation determining $x(t)$, we end up with a second-order linear differential equation that contains only $x(t)$ and its derivatives:

$$\ddot{x}(t) - (a_1 + b_2)\,\dot{x}(t) + (a_1 b_2 - b_1 a_2)\,x(t) = 0 \tag{2.19}$$

Equation (2.19) is a linear homogeneous second-order differential equation. It can be solved using the method of undetermined coefficients. Its characteristic equation is given by

$$r^2 - (a_1 + b_2)\,r + (a_1 b_2 - a_2 b_1) = 0 \tag{2.20}$$

If the roots of (2.20) are real and distinct, the solution of (2.19) is given by

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{2.21}$$

Solving the first equation of (2.18) with respect to $y(t)$, we get

$$y(t) = \frac{1}{b_1}\left(\dot{x}(t) - a_1 x(t)\right)$$

Substituting the solution (2.21) for $x(t)$ and its first derivative, we get

$$y(t) = \frac{1}{b_1} \left(\left(r_1 - a_1 \right) c_1 e^{r_1 t} + \left(r_2 - a_1 \right) c_2 e^{r_2 t} \right) \tag{2.22}$$

Consequently, the solution of the system (2.18) consists of equations (2.21) and (2.22), if the roots of (2.20) are real and distinct. We can solve the system in an analogous way if we have complex or repeated roots.

However, there is a second and more direct solution method of the homogeneous system (2.18). This method also generalizes to higher-order systems of differential equations.

2.4.2 The Method of Eigenvalues

Our experience with first-order differential equation suggests that we use the pair of equations

$$x(t) = Ae^{rt}, y(t) = Be^{rt}$$

as particular solutions for (2.18). Here A, B, and r are undetermined coefficients. Substituting these solutions in (2.18), we get

$$\begin{aligned} rAe^{rt} &= a_1 Ae^{rt} + b_1 Be^{rt} \\ rBe^{rt} &= a_2 Ae^{rt} + b_2 Be^{rt} \end{aligned} \tag{2.23}$$

Dividing both equations by e^{rt}, we can rewrite the system (2.23) in matrix form:

$$\begin{pmatrix} a_1 - r & b_1 \\ a_2 & b_2 - r \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.24}$$

For (2.24) to hold, the determinant of the matrix of coefficients must be zero:

$$\begin{vmatrix} a_1 - r & b_1 \\ a_2 & b_2 - r \end{vmatrix} = 0$$

Calculating the determinant, we get a quadratic equation in r :

$$r^2 - \left(a_1 + b_2 \right) r + \left(a_1 b_2 - a_2 b_1 \right) = 0 \tag{2.25}$$

Equation (2.25) is the characteristic equation of the system (2.18). Equation (2.25) is identical to (2.20), the equation we ended up with when using the method of substitution. The solutions of the characteristic equation (2.25) are called the eigenvalues of the matrix of coefficients:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

The two roots are given by

$$r_1, r_2 = \frac{(a_1 + b_2) \pm \sqrt{(a_1 + b_2)^2 - 4(a_1 b_2 - a_2 b_1)}}{2} \tag{2.26}$$

For future use, note that

$$\begin{aligned} r_1 + r_2 &= \quad a_1 + b_2 \\ r_1 r_2 &= a_1 b_2 - a_2 b_1 \end{aligned}$$

If the roots are real, and $r_1 \neq r_2$, then the general solution of the homogeneous system (2.18) is given by

$$\begin{aligned} x(t) &= A_1 e^{r_1 t} + A_2 e^{r_2 t}, \\ y(t) &= B_1 e^{r_1 t} + B_2 e^{r_2 t}, \end{aligned}$$

where A_1, A_2 are determined by boundary conditions; the roots are determined by (2.26); and B_1, B_2 are determined by (2.23) as

$$B_1 = \frac{r_1 - a_1}{b_1} A_1$$

$$B_2 = \frac{r_2 - a_1}{b_1} A_2$$

The solution is identical to (2.21) and (2.22). In the case of complex or repeated roots, the solution is analogous.

Having found the general solution to the homogeneous system (2.18), it remains to find a particular solution of (2.17), using, for example, the method of variation of parameters.

For the special case where p and g are constants, a special solution with constant x and y can be found by solving the system of equations

$$\begin{aligned} a_1 \bar{x} + b_1 \bar{y} + p = 0 \\ a_2 \bar{x} + b_2 \bar{y} + g = 0 \end{aligned} \tag{2.27}$$

Expressing (2.27) in matrix form and solving for $\bar{x}$ and $\bar{y}$, we get

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = - \begin{pmatrix} p \\ g \end{pmatrix}$$

The solution is then given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = - \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} p \\ g \end{pmatrix}$$

where $\bar{x}$ and $\bar{y}$ can be regarded as *steady state*, or equilibrium points. Whether the system converges globally to equilibrium depends on whether both roots are real and smaller than zero. In this case, the equilibrium is a *sink*. If both variables are predetermined, this is a *stable* equilibrium.

If we have a positive and a negative root, the equilibrium is called a *saddle point*. There is only **one unique path** that leads to this equilibrium, and this path is called the *saddle path*. The economy will converge to equilibrium if one variable is predetermined and the other not predetermined. The non-predetermined variable will jump to the unique adjustment path leading to equilibrium. Technically, the negative root corresponds to the predetermined variable (for which we solve backward), and the positive root corresponds to the non-predetermined variable (for which we solve forward).

Thus, a system with one predetermined and one non-predetermined variable has an equilibrium (a saddle point) if the matrix of coefficients has one positive and one negative eigenvalue.

2.4.3 A System of n First-Order Linear Differential Equations

Finally, we turn to a more general case with extensive applications in macroeconomics: a system of n first-order linear differential equations of the form

$$\begin{aligned} \dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + g_1(t) \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + g_2(t) \\ &\vdots \\ \dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + g_n(t) \end{aligned} \tag{2.28}$$

where $x_1, x_2, \ldots, x_n, a_{ij}$ for $i, j = 1, 2, \ldots, n$ are given constant parameters, and $g_1, g_2, \ldots, g_n$ are exogenous forcing functions of time.

In matrix form, this system can be written as

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) \tag{2.29}$$

where bold letters denote vectors and matrices; $\mathbf{A}$ is the square $n \times n$ coefficient matrix, which is assumed to be nonsingular.

2.4.4 Eigenvalues and Eigenvectors

Before we proceed to discuss the solution of the system of differential equations (2.28), it is worth delving a little more into linear algebra, and in particular, into the concepts of *eigenvalues* and *eigenvectors*.

Let $\mathbf{A}$ be a square matrix, like the one multiplying the $\mathbf{x}$ vector in the righthand side of (2.29). An eigenvalue of $\mathbf{A}$ is a number ρ, which when subtracted from each of the diagonal elements of $\mathbf{A}$ converts $\mathbf{A}$ into a singular (i.e., noninvertible) matrix. Subtracting a scalar ρ from each of the diagonal elements of $\mathbf{A}$ is equivalent to subtracting from $\mathbf{A}\rho$ times the identity matrix I. Therefore, ρ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A} - \rho\mathbf{I}$ is singular.

Because a matrix is singular if its determinant is equal to zero, ρ is an eigenvalue of $\mathbf{A}$ if and only if

$\det|\mathbf{A} - \rho\mathbf{I}| = 0.$

For an $n \times n$ matrix $\mathbf{A}$, the determinant of $\mathbf{A} - \rho\mathbf{I}$ is an n th-order polynomial in ρ, called the characteristic polynomial of $\mathbf{A}$. An n th-order polynomial has at most n roots. Therefore, an $n \times n$ square matrix has at most n eigenvalues.

It follows from the above that the diagonal entries of a diagonal matrix $\mathbf{D}$ are eigenvalues of $\mathbf{D}$, and that a square matrix $\mathbf{A}$ is singular if and only if 0 is an eigenvalue of $\mathbf{A}$.

Recall from elementary linear algebra that a square matrix $\mathbf{B}$ is nonsingular if and only if the only solution of $\mathbf{B}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Conversely, $\mathbf{B}$ is singular if and only if the system $\mathbf{B}\mathbf{x} = \mathbf{0}$ has a nonzero solution.

The fact that $\mathbf{A} - \rho\mathbf{I}$ is singular when ρ is an eigenvalue of $\mathbf{A}$ means that the system of equations $(\mathbf{A} - \rho\mathbf{I})\mathbf{v} = 0$ has a solution other than $\mathbf{v} = 0$.

When ρ is an eigenvalue of $\mathbf{A}$, a nonzero vector $\mathbf{v}$ such that $(\mathbf{A} - \rho\mathbf{I})\mathbf{v} = \mathbf{0}$, is called a (right) eigenvector of $\mathbf{A}$, corresponding to the eigenvalue ρ.

Thus, eigenvectors are nonzero vectors $\mathbf{v}$ that satisfy

$(\mathbf{A} - \rho\mathbf{I})\mathbf{v} = \mathbf{0}, \mathbf{A}\mathbf{v} - \rho\mathbf{I}\mathbf{v} = \mathbf{0}, \mathbf{A}\mathbf{v} = \rho\mathbf{v}.$

These three statements are equivalent.

2.5 Solving the n th-Order System of Linear Differential Equations

Let us now turn to the solution of the n th-order system of linear differential equations represented by (2.28). The general solution of the nonhomogeneous system of differential equations (2.28) will be the sum of the general solution of the relevant homogeneous system of differential equations plus the particular solution for constant x s.

We shall concentrate on the solution of the homogeneous system

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \tag{2.30}$$

or simply, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is the column vector on the right-hand side of (2.30).

Assume first that $\mathbf{A}$ is a diagonal matrix, for which $a_{ij} = 0$ for $i \neq j$. Then (2.30) becomes a system of n independent self-contained equations of the form

$$\dot{x}_i(t) = a_{ii}x_i(t)$$

We thus have a system of independent first-order linear differential equations that can be solved one by one:

$$x_i(t) = c_i e^{a_{ij}t}$$

If the off-diagonal elements a_{ij} differ from zero, so that the equations are linked to each other, then we can use the eigenvalues and eigenvectors of the coefficient matrix of (2.30) to transform it to a system of n (or fewer) independent equations. We can use the eigenvalues and eigenvectors of $\mathbf{A}$ to transform the system to a system that has a diagonal coefficient matrix.

Let us assume that $\mathbf{A}$ has n distinct real eigenvalues $\rho_1, \rho_2, \dots, \rho_n$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. It then follows from the definition of eigenvalues and eigenvectors that

$$\mathbf{A}\mathbf{v}_i = \rho_i \mathbf{v}_i, i = 1, 2, \dots, n \tag{2.31}$$

Let $\mathbf{P}$ be the $n \times n$ matrix whose columns are these n eigenvectors. Thus $\mathbf{P}$ is defined as

$$\mathbf{P} = [\mathbf{v}_1 \cdots \mathbf{v}_n] \tag{2.32}$$

The system of equations (2.31) can thus be written as

$$\mathbf{AP} = \mathbf{PJ}$$

where

$$\mathbf{J} = \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{pmatrix}$$

Because eigenvectors for distinct eigenvalues are linearly independent, $\mathbf{P}$ is nonsingular and therefore invertible. We can write

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{J}$$

Thus, we can use (2.32) to transform the system (2.28), defined in the variables $\mathbf{x}$, to a system in the variables $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, which means that $\mathbf{x} = \mathbf{Py}$. It follows that

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{APy} = \mathbf{Jy}$$

Because $\mathbf{J}$ is a diagonal matrix, the solution of the system (C.59) can be obtained very easily as the vector of solutions to each variable y_i :

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\rho_1 t} \\ \vdots \\ c_n e^{\rho_n t} \end{pmatrix}$$

Finally, we can use the transformation $\mathbf{x} = \mathbf{Py}$ to return to the original variables $x_1, \dots, x_n$:

$$\mathbf{x}(t) = \mathbf{Py}(t) = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{pmatrix} c_1 e^{\rho_1 t} \\ \vdots \\ c_n e^{\rho_n t} \end{pmatrix} = c_1 e^{\rho_1 t}\mathbf{v}_1 + c_2 e^{\rho_2 t}\mathbf{v}_2 + \cdots + c_n e^{\rho_n t}\mathbf{v}_n$$

Thus, under the assumption that the $n \times n$ matrix $\mathbf{A}$ has n distinct real eigenvalues $\rho_1, \rho_2, \dots, \rho_n$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, the general solution of the homogeneous linear system (2.28) is given by

$\mathbf{x}(t) = c_1 e^{\rho_1 t}\mathbf{v}_1 + c_2 e^{\rho_2 t}\mathbf{v}_2 + \cdots + c_n e^{\rho_n t}\mathbf{v}_n.$

The solution in the cases of complex eigenvalues or multiple eigenvalues without enough eigenvectors is analogous to the solution of the second-order homogeneous system analyzed in Section 2.4.2

Steady states and stability conditions are defined in an analogous way to those for first-and second-order differential equations. Assuming that the vector of g s consists of constants, one gets the nonhomogeneous system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

The steady state, if it exists, can be derived by setting the change in the x s equal to zero. A steady state exists if $\mathbf{A}$ is nonsingular and is given by

$$\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} = - \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

where $\bar{x}_i$ denotes the steady state value of x_i. These values can be regarded as equilibrium points. If the x_i s are predetermined variables, for the system to converge to equilibrium, all eigenvalues must be less than zero. In this case, the equilibrium is a fixed node. When the x_i s consist of p predetermined and q non-predetermined variables, where $p + q = n$, the equilibrium (if it exists) is a saddle point. For the system to converge to equilibrium, there must be p negative eigenvalues and q positive eigenvalues. The negative eigenvalues correspond to the predetermined variables, which are solved for backward, and the positive eigenvalues correspond to the non-predetermined variables which are solved for forward.

Thus, a system with p predetermined and q non-predetermined variables has a stable equilibrium (a saddle vector) if the matrix of coefficients has p negative and q positive eigenvalues. The adjustment path is unique and is called a saddle path.

2.6 Nonlinear Differential Equations

In this section, we consider problems of the form:

$$\dot{x}(t) = f(x(t)) \tag{2.33}$$

where $x(t)$ is a vector $x(t) = (x_1(t), x_2(t), ...x_n(t))$ and f is a shortcut notation for n different functions. This kind of system may be difficult (or even impossible) to solve. However, it is possible to derive some approximation of the solution by taking a first-order expansion (or eventually higher-order) around a reference point. A good candidate is the steady state $\bar{x}$ satisfying $f(\bar{x}) = 0$. Hence, we get:

$$d\dot{x}(t) = \underbrace{f(\bar{x})}_{=0} + J(\bar{x}) \cdot (x(t) - \bar{x})$$

where $J(\bar{x})$ is the Jacobian matrix with entries $f'(\bar{x})$ for each $x_i(t)$,$i = 1, ..., n$.

Treating $(x(t) - \bar{x}) = dx(t)$ as the new variable of interest, one can see that we obtain a n system of first-order linear differential equations and use the techniques from Section 2.5.

2.7 Qualitative Analysis

It is possible to introduce some qualitative informations about systems. We focus on a generic pair of linear differential equation of the form:

$$\dot{x}(t) = ax_t + by_t + c$$
$$\dot{y}(t) = dx_t - ey_t + f$$

Without loss of generality, assume that all parameters a, b, c, d, e, f are positive. We can draw the isoclines $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$ in the plane $\{x(t), y(t)\}$, that is the *loci*:

$$\dot{x}(t) = 0: \quad \tilde{y} = \frac{-ay + c}{b}$$
$$\dot{y}(t) = 0: \quad \hat{y} = \frac{dx + f}{e}$$

The next step is to characterize the *vector field*, that is the qualitative changes of x_t and y_t for every pair x, y outside the isoclines. For $\dot{x}(t) = 0$, we have:

$$\dot{x}(t) = 0 \geq 0 \Leftrightarrow y(t) \geq \frac{-ax(t) + c}{b} \equiv \tilde{y}$$
$$\dot{y}(t) = 0 \leq 0 \Leftrightarrow y(t) \leq \frac{dx(t) + f}{e} \equiv \hat{y}$$

On the first hand, for every pair $\{x(t), y(t)\}$ located on the right (left) of the locus $\tilde{y}$, the value of $x(t)$ increases (decreases) and goes on the right (left). On the other hand, for every pair $\{x(t), y(t)\}$ located below (above) the locus $\hat{y}$, the value of $y(t)$ increases (decreases) and goes upward (downward).

3 Difference Equations

In this chapter, we review the properties and solution methods of first- and second-order difference equations, as well as of systems of first-order difference equations.

Difference equations are the analog of differential equations when time is a discrete variable defined in terms of integers. They are an indispensable tool for the study of dynamic economic problems in discrete time. We shall thus assume that time is an integer $t = \ldots, -2, -1, 0, 1, 2, \ldots$, instead of t being a real continuous variable.

After defining lag operators, we then proceed to present solution methods for first- and second-order linear difference equations and for systems of interdependent linear difference equations.

3.1 Lag Operators and Difference Equations

To define and analyze difference equations, it is useful to first define lag operators. The value of a variable x in period t is denoted by x_t. The lag operator L for a variable x_t is defined by

$$L^n x_t = x_{t-n}$$

for $n = \ldots, -2, -1, 0, 1, 2, \ldots$

Thus, the multiplication of x_t by L denotes the value of the variable in the previous period, and the multiplication of the variable by L^n denotes the value of the variable in period $t - n$. Note than if n is negative (i.e., $n < 0$), the lag operator shifts the variable n periods into the future.

This definition is mathematically somewhat loose. More formally, let us assume a sequence

$$\{x_t\}_{t=-\infty}^{\infty}$$

that links a real number x with every integer t. Applying the operator L^n to this sequence, we get a new sequence:

$$\{y_t\}_{t=-\infty}^{\infty} = \{x_{t-n}\}_{t=-\infty}^{\infty}$$

The operator L^n projects one sequence onto another.

Let us now examine a polynomial in the lag operator:

$$A(L) = a_0 + a_1 L + a_2 L^2 + \cdots = \sum_{j=0}^{\infty} a_j L^j$$

Applying the polynomial $A(L)$ to variable x_t, we get a moving sum of x s in different time periods:

$$A(L)x_t = \sum_{j=0}^{\infty} a_j L^j x_t = \sum_{j=0}^{\infty} a_j x_{t-j}$$

Let us confine ourselves to rational functions (i.e., polynomials) that can be expressed as the ratio of two finite polynomials in L. Assume that

$$A(L) = \frac{B(L)}{C(L)} \tag{3.1}$$

where

$$\begin{aligned} B(L) &= \sum_{j=0}^{m} b_j L^j \\ C(L) &= \sum_{j=0}^{n} c_j L^j \end{aligned} \tag{3.2}$$

and b_j and c_j are constants. The combination of (3.1) and (3.2) imposes a more economical and restrictive form on a_j, without serious loss of generality.

A special case of (3.1) and (3.2) is the so-called geometric polynomial, which takes the form

$$A(L) = \frac{1}{1 - \lambda L} \tag{3.3}$$

From the properties of geometric progressions, the geometric polynomial can be expanded in two ways:

$$A(L) = \frac{1}{1 - \lambda L} = 1 + \lambda L + \lambda^2 L^2 + \cdots \tag{3.4}$$

$$A(L) = \frac{1}{1-\lambda L} = -\frac{1}{\lambda L}\left(1 + \frac{1}{\lambda}L^{-1} + \frac{1}{\lambda^2}L^{-2} + \cdots\right) \tag{3.5}$$

The expansion (3.4) is used when $|\lambda| < 1$, and the expansion (3.5) when $|\lambda| > 1$.

If we multiply the geometric polynomial (3.3) by some variable x_t, we get

$$A(L)x_t = \frac{1}{1-\lambda L}x_t \tag{3.6}$$

With the expansion (3.4) for $A(L)$ we get

$$A(L)x_t = \frac{1}{1-\lambda L}x_t = \sum_{i=0}^{\infty} \lambda^i L^i x_t = \sum_{i=0}^{\infty} \lambda^i x_{t-i} \tag{3.7}$$

If $|\lambda| < 1$, and $\{x_t\}_{l=-\infty}^{\infty}$ is a finite sequence of real numbers, then (3.7) defines a finite sequence of real numbers as well.

In contrast, we have the alternative expansion of (3.6). Using (3.5), we get

$$A(L)x_t = \frac{1}{1-\lambda L}x_t = (\lambda L)^{-1} \sum_{i=0}^{\infty} \lambda^{-i} L^{-i} x_t = \sum_{i=0}^{\infty} \lambda^{-i} x_{t+i} \tag{3.8}$$

If $|\lambda| > 1$, and $\{x_t\}_{t=-\infty}^{\infty}$ is a finite sequence of real numbers, then (3.8) defines a finite sequence of real numbers as well, because we have $\left|\lambda^{-1}\right| < 1$.

In economics, because we usually seek convergence to some equilibrium, we seek the analysis of finite sequences. Thus, we select the backward expansion when $|\lambda| < 1$, and the forward expansion when $|\lambda| > 1$.

A difference equation (or recurrence relation) equates a polynomial in the various iterates of a variable-that is, in the values of the elements of a sequence-to zero.

An n th-order linear difference equation with constant coefficients takes the form

$$a_0 x_t + a_1 x_{t-1} + a_2 x_{t-2} + \cdots + a_n x_{t-n} - b = \sum_{j=0}^{n} a_j L^j x_t - b = 0$$

where $a_j, j = 0, 1, 2, \dots, n$ and b are constant coefficients.

By equating the right-hand side of the geometric polynomial (3.7) to zero, we get

$$\sum_{i=0}^{\infty} \lambda^i x_{t-i} = x_t + \lambda x_{t-1} + \lambda^2 x_{t-2} + \cdots = 0$$

This is an example of an infinite-order linear difference equation.

3.2 First-Order Linear Difference Equations

Let us first consider the first-order linear difference equation with constant coefficients:

$$x_t = a + b x_{t-1} \tag{3.9}$$

Using lag operators, (3.9) can be written as

$$(1 - bL)x_t = a \tag{3.10}$$

Dividing both sides of (3.10) by $(1 - bL)$ and adding cb^t, we get

$$x_t = \frac{a}{1 - bL} + cb^t = \frac{a}{1 - b} + cb^t \tag{3.11}$$

where c is an arbitrary constant. We include the term cb^t because for any c,

$$(1 - bL)cb^t = cb^t - bcb^{t-1} = 0$$

Hence, if we multiply (3.11) by $(1 - bL)$, we get back (3.10). Equation (3.11) determines the general solution of the linear first-order difference equation (3.9).

To find a particular solution, we must determine c. Assume that in period $t = 0, x$ had the value x_0. From (3.11), it follows that

$$c = x_0 - \frac{a}{1 - b}$$

Thus, the particular solution of (3.9) is given by

$$x = \frac{a}{1 - b} + \left(x_0 - \frac{a}{1 - b}\right) b^t \tag{3.12}$$

If the boundary condition is such that $x_0 = a/(1 - b)$, then (3.12) implies that

$$x_t = x_0, \forall t \geq 0$$

Thus, $a/(1-b)$ can be seen as an equilibrium value. If $x = a/(1-b)$, then x tends to stay at this level.

In addition, x_0 if $|b| < 1$, (3.12) implies that for any x_0, we have

$$\lim_{t \to \infty} x_t = \frac{a}{1-b} \tag{3.13}$$

Equation (3.13) implies that the difference equation is stable, because x tends to approach its equilibrium value over time from any initial condition. In this case, the equilibrium value is a stable node.

If $|b| > 1$, the only path that leads to the equilibrium value is the immediate jump of x to the equilibrium value $a/(1-b)$. This solution requires

$c = 0, x_t = a/(1-b), \forall t.$

The equilibrium value in this case is a saddle point.

3.3 Second-Order Linear Difference Equations

We next turn to the second-order linear difference equation with constant coefficients, of the form

$$x_t = a + bx_{t-1} + cx_{t-2} \tag{3.14}$$

Using the lag operator, (3.14) can be written as

$$\left(1 - bL - cL^2\right) x_t = a \tag{3.15}$$

Equation (3.15) can be expressed as

$$\left(1 - \lambda_1 L\right)\left(1 - \lambda_2 L\right) x_t = a \tag{3.16}$$

where

$$\lambda_1 + \lambda_2 = b$$

$$\lambda_1\lambda_2 = -c$$

and λ_1 and λ_2 are the two roots of the second-order linear difference equation (3.14).

There are three possible cases, depending on the discriminant of the characteristic equa-tion of (3.14).

- Case 1: $b^2 > -4c$ The discriminant is positive, and the roots are real and distinct, taking the form

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4c}}{2}$$
$$\lambda_2 = \frac{b - \sqrt{b^2 + 4c}}{2}$$

From (3.16), the general solution of (3.14) takes the form

$$x_t = \frac{a}{(1-\lambda_1)(1-\lambda_2)} + d_1\lambda_1^t + d_2\lambda_2^t = \frac{a}{(1-b-c)} + d_1\lambda_1^t + d_2\lambda_2^t$$

where d_1 and d_2 are two arbitrary constants. To determine the arbitrary constants, one needs two boundary conditions, depending on the values of the two roots.

As in the case of a first-order difference equation, $a/(1-b-c)$ can be seen as the equili-brium value of x.

We have convergence to the equilibrium value if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. In this case, the equilibrium value will be a stable node, and to determine the two arbitrary constants, d_1 and d_2, we need two initial conditions $x_1, x_2 \neq 0$.

If the two roots lie on either side of unity (i.e., if $|\lambda_1| < 1$ and $|\lambda_2| > 1$), then the equilibrium value will be a saddle point. In this case, to determine the two arbitrary constants d_1 and d_2, we need one initial and one final condition. The final condition can be none other than the equilibrium value. As a result, we shall have convergence to the equilibrium value only if $d_1 \neq 0$ and $d_2 = 0$.

If both roots are greater than unity (i.e., if $|\lambda_1| > 1$ and $|\lambda_2| > 1$), then the only solution is the immediate jump of x to the equilibrium value $a/(1-b-\ c)$. This solution requires $d_1 = 0, d_2 = 0$, and $x_t = a/(1-b-c)$ for all t.

- Case 2: $b^2 = -4c$ The discriminant is equal to zero, and we have two equal real roots of the form

$$\lambda_1 = \lambda_2 = \lambda = \frac{b}{2}.$$

The general solution takes the form

$$x_t = \frac{a}{(1-b-c)} + d_1\lambda^t + d_2 t\lambda^t$$

If $|\lambda| < 1$, to determine the two arbitrary constants d_1 and d_2, we need two initial conditions.

If λ is greater than unity in absolute value (i.e., if $|\lambda| > 1$), then the only solution is the immediate jump of x to the equilibrium value $a/(1-b-c)$. This solution requires $d_1 = 0, d_2 = 0$, and $x_t = a/(1-b-c)$ for all t.

- Case 3 $b^2 < -4c$ The discriminant is negative, and we have two complex roots, which take the form of a pair of complex conjugates:

$$\lambda_1 = \mu + \nu i$$

$$\lambda_2 = \mu - vi$$

where $\mu = \frac{b}{2}$, and $v = \frac{\sqrt{-4c-b^2}}{2}$.

Using De Moivre's theorem and the Pythagorean theorem the solution takes the form

$$x_t = \frac{a}{1-b-c} + R^t\left((d_1+d_2)\cos(\theta t) + (d_1-d_2)\sin(\theta t)\right)$$

where R and θ are defined by

$$R = \sqrt{\mu^2+v^2} = \sqrt{\frac{b^2-4c-b^2}{4}} = \sqrt{-c}$$

and

$$\cos(\theta) = \frac{\mu}{\sqrt{-c}} = \frac{b}{2\sqrt{-c}}, \quad \sin(\theta) = \frac{v}{\sqrt{-c}} = \frac{\sqrt{-4c-b^2}}{2\sqrt{-c}}$$

This solution will produce oscillations of a periodic nature. The oscillations will be dampened if and only if $|c| < 1$. In such a case, there will be cyclical convergence to the equilibrium value. In the case $|c| < 1$, there will be continuous oscillations of a constant periodicity. And if $|c| > 1$, there will be divergent oscillations, unless x jumps immediately to its equilibrium value.

3.4 A Pair of First-Order Linear Difference Equations

We next turn to a second-order system of two linear first-order difference equations. The system is described by

$$\begin{aligned} x_t &= a_{10} + a_{11} x_{t-1} + a_{12} y_{t-1} \\ y_t &= a_{20} + a_{21} x_{t-1} + a_{22} y_{t-1} \end{aligned} \tag{3.17}$$

As in the case of a system of two first-order differential equations, the first method of solving this system is the substitution method. We can use the second equation to substitute for y_{t-1} in the first equation, and thus obtain a second-order difference equation in x :

$$x_t = a + b x_{t-1} + c x_{t-2} \tag{3.18}$$

where $a = (a_{10}(1 - a_{22}) + a_{12} a_{20}), b = a_{11} + a_{22}$, and $c = -(a_{11} a_{22} - a_{12} a_{21})$.

Equation (3.18) has the same form as (3.14) and can be solved as an ordinary second-order linear difference equation with constant coefficients. Making use of the lag operator, the homogeneous equation corresponding to (3.18) can be written as

$$\left(L^2 - \frac{a_{11} + a_{22}}{a_{11} a_{22} - a_{12} a_{21}} L + \frac{1}{a_{11} a_{22} - a_{12} a_{21}} \right) x_t = 0.$$

The two roots of the polynomial in the lag operator must satisfy the characteristic equation

$$\lambda^2 - \frac{a_{11} + a_{22}}{a_{11} a_{22} - a_{12} a_{21}} \lambda + \frac{1}{a_{11} a_{22} - a_{12} a_{21}} = 0 \tag{3.19}$$

By going through the alternative substitutions, a similar second-order difference equation can be obtained for the second variable y_t.

Alternatively, one can rewrite the system (3.17) in matrix form as

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} \tag{3.20}$$

The homogeneous system corresponding to (3.20), with $a_{10} = a_{20} = 0$, takes the form

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} \tag{3.21}$$

Using the lag operator L, (3.20) can be rewritten as

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \left(\begin{array}{cc} 1 - a_{11}L & -a_{12}L \\ -a_{21}L & 1 - a_{22}L \end{array} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For (3.20) to have a solution, the matrix in the lag operator must be singular. Therefore, its determinant must be equal to zero. Thus, for a solution to exist, we must have

$$\det \left(\begin{array}{cc} 1 - a_{11}L & -a_{12}L \\ -a_{21}L & 1 - a_{22}L \end{array} \right) = 0$$

This condition implies a polynomial in the lag operator with characteristic equation

$$\lambda^2 - \frac{a_{11} + a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\lambda + \frac{1}{a_{11}a_{22} - a_{12}a_{21}} = 0$$

which is identical to (3.19), the characteristic equation of the second-order difference equation (3.18), and will of course give the same solution for the two roots.

However, even this solution method becomes unwieldy for higher-order systems when there are more than two variables. It is thus desirable to investigate other solution methods. To do so, it is worth generalizing the system to one of n first-order linear difference equations.

3.5 A System of n First-Order Linear Difference Equations

Let us consider the following system of n first-order difference equations. Such systems arise quite often in dynamic macroeconomics:

$$\begin{aligned} x_{1,t} &= a_{10} + a_{11}x_{1,t-1} + a_{12}x_{2,t-1} + \cdots + a_{1n}x_{n,t-1} \\ x_{2,t} &= a_{20} + a_{21}x_{1,t-1} + a_{22}x_{2,t-1} + \cdots + a_{2n}x_{n,t-1} \\ &\vdots \\ x_{n,t} &= a_{n0} + a_{n1}x_{1,t-1} + a_{n2}x_{2,t-1} + \cdots + a_{nn}x_{n,t-1} \end{aligned} \tag{3.22}$$

In matrix form, the system (3.22) can be written as

$$\left(\begin{array}{c} x_{1,t} \\ \vdots \\ x_{n,t} \end{array} \right) = \left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right) \left(\begin{array}{c} x_{1,t-1} \\ \vdots \\ x_{n,t-2} \end{array} \right) + \left(\begin{array}{c} a_{10} \\ \vdots \\ x_{n0} \end{array} \right) \tag{3.23}$$

By defining the vector of x as $\mathbf{x}$, the matrix of multiplicative parameters as $\mathbf{A}$, and the vector of the constants as $\mathbf{a}_0$, the system can be written as

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{a}_0 \tag{3.24}$$

If $a_{ij} = 0$ for $i \neq j$ in the system (3.23), then the n equations are uncoupled, and the system can be solved as n independent first-order linear difference equations with solutions

$$x_{i,t} = \frac{a_{i0}}{1 - a_{ii}} + \left(x_{i,0} - \frac{a_{i0}}{1 - a_{ii}}\right)(a_{ii})^t$$

where $x_{i,0}$ is a boundary value for x_i.

Thus, if we could transform the system (3.23) into one with a coefficient matrix that is diagonal, we could easily calculate the solution. The question is how to transform the system into one with a diagonal coefficient matrix.

We know from the properties of matrices that the coefficient matrix $\mathbf{A}$ can be transformed as

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1} \tag{3.25}$$

where $\mathbf{J}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on its diagonal, and $\mathbf{P}$ is a matrix consisting of the corresponding (right) eigenvectors. Equation (3.25) implies that

$$\mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{J}$$

We can use these properties to rewrite the system (3.24) as

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{a}_0 = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}\mathbf{x}_{t-1} + \mathbf{a}_0 \tag{3.26}$$

Multiplying both sides of (3.26) by $\mathbf{P}^{-1}$, we get

$$\mathbf{P}^{-1}\mathbf{x}_t = \mathbf{J}\mathbf{P}^{-1}\mathbf{x}_{t-1} + \mathbf{P}^{-1}\mathbf{a}_0$$

Defining a new vector of variables $\mathbf{X}_t = \mathbf{P}^{-1}\mathbf{x}_t$, and a new vector of constants $\mathbf{A}_0 = \mathbf{P}^{-1}\mathbf{a}_0$, we can rewrite (3.24) as

$$\mathbf{X}_t = \mathbf{J}\mathbf{X}_{t-1} + \mathbf{A}_0$$

Thus, by using the matrix consisting of the eigenvectors of the original coefficient matrix $\mathbf{A}$ to define new variables and new constants, we can transform the original coupled system of difference equations to a system of decoupled difference equations in the newly defined variables, as in the case of differential equations. The decoupled system can be solved for each of the transformed variables in $\mathbf{X}$. We can then find the solutions for the original vector of variables by using the reverse transformation

$$\mathbf{x}_t = \mathbf{P}\mathbf{X}_t$$

By using the diagonal matrix of the eigenvalues of $\mathbf{A}$ and the matrix of the corresponding (right) eigenvectors, we can solve the system of n first-order linear difference equations (3.22). The solution method is similar in spirit to the one for a system of n first-order differential equations discussed in Chapter 2.

! Important

Keep in mind that the conditions for stability for discrete and continuous time systems are different. In continuous time, the conditions requires to compare the eigenvalues λ_i to zero (i.e. a positive eigenvalue is unstable, a negative eigenvalue is stable). In contrast, in discrete time systems, the eigenvalues λ_i (in absolute value) has to be compared to unity.(i.e. a stable eigenvalue is within the unit circle, an unstable eigenvalue is outside the unit circle).

3.6 Nonlinear difference equations

We are interested now in problems of the form:

$$x_{t+1} = f(x_t)$$

with x_t a vector $x_t = (x_{1,t}, x_{2,t}, ..., x_{n,t})$ and f is a shortcut notation for n different functions. Just like in the case of differential equations, we can approximate in the neigborhood of the steady state $\bar{x}$ satisfying $\bar{x} = f(\bar{x})$. Taking a first-order expansion:

$$x_{t+1} = \underbrace{f(\bar{x})}_{=\bar{x}} + J(x_t - \bar{x})$$

Where J is the Jacobian matrix with entries $f'(\bar{x})$ for each $x_{i,t}$, $i = 1, ..., n$.

Treating $dx_t = (x_t - \bar{x})$ and $dx_{t+1} = (x_{t+1} - \bar{x})$ as the new variables of interest, one can see that we obtain a n system of first-order linear differential equations and use the techniques from Section 3.5.

3.7 Qualitative Analysis

It is possible to introduce some qualitative informations about systems. We focus on a generic pair of linear difference equation of the form:

$$x_{t+1} = \tilde{a}x_t + by_t + c$$
$$y_{t+1} = dx_t - \tilde{e}y_t + f$$

Define $\Delta x_{t+1} = x_{t+1} - x_t$ and $\Delta y_{t+1} = y_{t+1} - y_t$, the above system rewrite:

$$\Delta x_{t+1} = ax_t + by_t - c :$$
$$\Delta y_{t+1} = dx_t - ey_t + f$$

where $a = \tilde{a} - 1$ and $e = 1 + \tilde{e}$. Without loss of generality, assume that all parameters a, b, c, d, e, f are positive. We can draw the isoclines $\Delta x_{t+1} = 0$ and $\Delta y_{t+1} = 0$ in the plane $\{x_t, y_t\}$, that is the *loci*:

$$\Delta x_{t+1} = 0 : \quad \tilde{y} = \frac{-ax + c}{b}$$
$$\Delta y_{t+1} = 0 : \quad \hat{y} = \frac{dx + f}{e}$$

The next step is to characterize the *vector field*, that is the qualitative changes of x_t and y_t for every pair x, y outside the isoclines. For Δx_{t+1}, we have:

$$\Delta x_{t+1} \geq 0 \Leftrightarrow y_t \geq \frac{-ay + c}{b} \equiv \tilde{y}$$
$$\Delta y_{t+1} \leq 0 \Leftrightarrow y_t \leq \frac{dx + f}{e} \equiv \hat{y}$$

On the first hand, for every pair $\{x_t, y_t\}$ located on the right (left) of the locus $\tilde{y}$, the value of x_t increases (decreases) and goes on the right (left). On the other hand, for every pair $\{x_t, y_t\}$ located below (above) the locus $\hat{y}$, the value of y_t increases (decreases) and goes upward (downward).

4 Intertemporal Optimization

4.1 Introduction

In this chapter, we study techniques applied to dynamic optimizations. Optimization in dynamics economic problems, which are problems in which variables change over time, does not requires new principles vis-à-vis static problems but possesses a specific structure needed to take care about.

The most important part of this specific structure is the relation between *stocks* and *flows*. Some variables, which we will denote by y, have the form of stocks, changing gradually over time. Other variables, which we will denote by x, have the form of flows, which can change freely at any instant. Mathematically, stocks are called *state variables* and flows are called *control variables*.[1]

Stock variables evolves according both stocks and flows, control variables control changes in state variables. For instance, savings in period t determine the change in households wealth from period t to period $t+1$. A general form of the volution of state variables is:

$$y_{t+1} - y_t = Q(y_t, x_t, t) \tag{4.1}$$

where $t, t+1, ...$ are discrete time periods, y is a state variable (stocks), x is a control variable (flows) and Q is a vector function. There might be additional restrictions that we summarize under the form:

$$G(y_t, x_t, t) \leq 0 \tag{4.2}$$

Note that restrictions (4.1) and (4.2) have different structure: the former involves directly a dynamic restriction while the latter is a static.

Furthermore, in most dynamic economic problems, agents have to optimize an objective function of the following additively-separable form:

$$\sum_{t=0}^{T} F(y_t, x_t, t) \tag{4.3}$$

[1] Not all state variables are stock variable. For instance, past decisions on controls can also be considered as state.

subject to restrictions (4.1) and (4.2). Periods start at $t = 0$ and end at T, which is potentially infinite. For instance, the function $F(y_t, x_t, t)$ can represent some households' utility function or firms' profit at date t.[2] Hence, we seek, in such examples, to maximize the sum of stream of instantaneous utilities (or profits) over time. The value of the initial stock at time 0 is taken as given but also at date T+1: such problems have therefore both initial and terminal conditions, the terminal condition being called *transversality conditions*.

We are going to present two widely used techniques in economics: optimal control and dynamic programming. Both are going to be applied in discrete and continuous time.

4.2 Optimal Control Methods

4.2.1 The Optimal Control Method in Discrete Time

In the optimal control problem, we want to select the variables y_t and x_t for $t = 0, 1, 2..., T$ to find the optimal solution of (4.3) subject to the constraints (4.1) and (4.2). To say it differently, we want to find the sequence $x_0, x_1, x_2 ... x_T$ and $y_1, y_2, ..., y_T$ (remember that y_0 and y_{T+1} are given) satisfying this problem.

To do so, we define multipliers (or shadow values) and construct the Lagrangian function. Define as μ_t the multiplier for the constraints (4.2). These have the usual interpretation of shadow values for the constraints in period t. The multipliers for the constraints (4.1) are different since they define the first-order change in the objective function if the constraint in the change of the stock is loosened (i.e. if we have a marginal increase in the stock variable y_{t+1}). They are therefore shadow values of the stock variables in period $t + 1$ and we denote them λ_{t+1}.

We define as $\mathcal{L}$ the Lagrangian function of the full intertemporal problem:

$$\mathcal{L} = \sum_{t=0}^{T} \{F(y_t, x_t, t) + \lambda_{t+1} [y_t + Q(y_t, x_t, t) - y_{t+1}] - \mu_t G(y_t, x_t, t)\}$$

Note that because of the time additive structure of the problem, multipliers are also dated at some period.

The first-order conditions for the optimization of the Lagrangian function with respect to the control variables x are:

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 : \quad F_x(y_t, x_t, t) + \lambda_{t+1} Q_x(y_t, x_t, t) - \mu_t G_x(y_t, x_t, t) = 0$$

[2] In such problems, we may see x_t and y_t as respectively the consumption level and the wealth owned at date t in the case of a household, or for the problem of a firm, x_t as the investment decision and y_t as the capital stock installed.

for $t = 0, 1, ..., T$ and where F_x, Q_x, and G_x are the partial first-order derivatives with respect to x. Note again that we have $T + 1$ first-order conditions with respect to x but the only real differences between them is their date.

With respect to y, the first-order conditions are a bit more complex because y appears in two consecutive periods, and therefore two terms of the sum. Let's rearrange the Lagrangian function to see it clearly.

$$\begin{aligned}\mathcal{L} = \sum_{t=1}^{T} \{F(y_t, x_t, t) + \lambda_{t+1} Q(y_t, x_t, t) + y_t(\lambda_{t+1} - \lambda_t) - \mu_t G(y_t, x_t, t)\} \\ + F(y_0, x_0, 0) + \lambda_1 Q(y_0, x_0, 0) + y_0 \lambda_1 - y_{T+1} \lambda_{T+1}\end{aligned}$$

The final four terms in the previous equation refer to given value of y in period 0 and $T + 1$. The first order-condition for the optimum of $\mathcal{L}$ for y_t, $t =, 1, 2..., T$ are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_t} = 0: \quad & F_y(y_t, x_t, t) + \lambda_{t+1} Q_y(y_t, x_t, t) + \lambda_{t+1} - \lambda_t - \mu_t G_y(y_t, x_t, t) = 0 \\ & \Rightarrow \lambda_{t+1} - \lambda_t = -\left[F_y(y_t, x_t, t) + \lambda_{t+1} Q_y(y_t, x_t, t) - \mu_t G_y(y_t, x_t, t)\right]\end{aligned} \tag{4.4}$$

These conditions can be written in a more comprehensive and economically useful way. Define a new function, the Hamiltonian function $\mathcal{H}$ as follows:

$$\mathcal{H}(y_t, x_t, \lambda_{t+1}, t) = F(y_t, x_t, t) + \lambda_{t+1} Q(y_t, x_t, t) \tag{4.5}$$

Equation (4.5) suggests that the control variable x must be selected to optimize $\mathcal{H}(y_t, x_t, \lambda_{t+1}, t)$ under the constraint $G(y_t, x_t, t) \leq 0$ which can be written into a new Lagrangian:

$$\tilde{\mathcal{L}}(y_t, x_t, \lambda_{t+1}, t) = \mathcal{H}(y_t, x_t, \lambda_{t+1}, t) - \mu_t G(y_t, x_t, t)$$

Then equation (4.4) can be written more simply as:

$$\lambda_{t+1} - \lambda_t = -\tilde{\mathcal{L}}_y(y_t, x_t, \lambda_{t+1}, t) \tag{4.6}$$

Finally, from (4.1) and (4.5), we get (using the Enveloppe Theorem):

$$y_{t+1} - y_t = \tilde{\mathcal{H}}_\lambda(y_t, x_t, \lambda_{t+1}, t) = Q(y_t, x_t, t) \tag{4.7}$$

These properties of the Hamiltonian function are known as the **Maximum Principle**:

Maximum Principle

The necessary first-order conditions for the optimization of (4.3) under the constraints (4.1) and (4.2) are the following: (1) for each t, the control variables x_t optimize the Hamiltonian function (4.5) under the static constraints (4.2). (2) The changes of y_t and λ_t over time are determined by the difference equations (4.6) and (4.7).

The Maximum Principle, proposed by Pontryagin *et al.* (1962), facilitates the determination of the first-order conditions for intertemporal optimization problems. It also gives easier interpretations of the first-order conditions of dynamic economic problems we have just seen.

In particular, changes in the decision variables x_t directly impact the objective function (4.3) but also on y_{t+1} through the impact on Q. Hence, the change in the objective function is found by multiplying the impact of x on Q with the shadow value λ_{t+1} of y_{t+1}. The Hamiltonian provides a simple way of converting the one-period objective function F to account for the future impact of the current choice of the control variable x. A similar economic interpretation can be obtained to the first-order conditions for the state variable y. A marginal change in y in period t gives the marginal change $F_y - \mu G_y$ in period t, given the shadow value of λ_{t+1} The right-hand side of (4.4) may be interpreted as a dividend. The change $\lambda_{t+1} - \lambda_t$ is like a capital gain. Equation (4.4) tells us that he dividend plus the capital gain should be equal to zero: at the optimim, there can be no excess return from y.

4.2.2 The Optimal Control Method in Continuous Time

In many applications, it is more convenient to treat time as a continuous variable. In such case, Equations (4.1) and (4.5) become:

$$\dot{y}(t) = \frac{dy}{dt} = Q(y(t), x(t), t) \tag{4.8}$$

$$G(y(t), x(t), t) \leq 0 \tag{4.9}$$

For the objective function, Equation (4.3) in continuous time becomes:

$$\int_0^T F(y(t), x(t), t)dt \tag{4.10}$$

We can use the Hamiltonian function as before:

$$\mathcal{H} = F(y(t), x(t), t) + \lambda(t)Q(y(t), x(t), t) \tag{4.11}$$

The FOC for the optimization of the Hamiltonian function (4.11) under the static constraints (4.9) are:

$$\frac{\partial \mathcal{H}}{\partial x(t)} - \mu(t)\frac{\partial G}{\partial x(t)} = 0 \Rightarrow F_x + \lambda(t)Q_x = \mu(t)G_x \tag{4.12}$$

$$\frac{\partial \mathcal{H}}{\partial y(t)} = -\dot{\lambda}(t) \Rightarrow F_y + \lambda(t)Q_y = -\dot{\lambda}(t) \tag{4.13}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda(t)} = \dot{y}(t) \Rightarrow \dot{y}(t) = Q(y(t), x(t), t) \tag{4.14}$$

These three equations are the continuous-time equivalent of the discrete time first-order conditions previously obtained.

4.3 Dynamic Programming and the Bellman Equation

Dynamic programming is an alternative method of solving the problem at the beginning of this chapter. It is an extremely useful in problems that combine time and uncertainty as ofent happens in economics.

Our problem is the optimization of:

$$\sum_{t=0}^{T} F(y_t, x_t, t)$$

under the constraints:

$$y_{t+1} - y_t = Q(y_t, x_t, t)$$

and

$$G(y_t, x_t, t) \leq 0$$

for $t = 0, 1, 2, .., T$. Again, the vectors of initial and final stocks y_0 and y_{T+1} are taken as given. We can define the optimal value that comes out of this problem as a function of the initial stocks y_0. Denote this function as $V(y_0)$. The vector of the first derivatives of this function $V_y(y_0)$ is the vector of the shadow values of these initial stocks.

The additive separability of the objective function and the constraints allow us to make an important generalization of the above ide. Instead of starting off at time zero, let us assume that we start off at time $t = \tau$. For the decisions thaty start at τ, the only thing that matters from the past is the vector of stocks y_τ, which is the result of past decisions. Our problem is to optimize an objective function such as (4.3) and the associated constraints, with time starting from τ and not 0. We define $V(y_\tau, \tau))$ as the optimal value that emerges as a function of stocks y_τ and period τ. The vector of the first derivatives $V_y(y_\tau, \tau)$ denotes the marginal increase in the optimal value for a small increase of stocks in period τ, which is the vector of shadow values of the initial stocks for the optimization problem that starts in period τ. This applies at all t.

Let us then select any t and exime the decision of choosing the values of the control variables for thaht period. Any choice of the control viarable x_t will lead to stocks y_{t+1} through ([-#eq-generaldynamics]). What remains is to solve the subproblem for period $t+1$ and to find the optimal value $V(y_{t+1}, t+1)$. The total value in period t of a choice for the control variables x_t starting oof with stocks y_t, can be separated into two terms: $F(y_t, x_t, t)$ which occurs in the current period; and $V(y_{t+1}, t+1)$, which comes about in future periods. The choice of x_t must optimize the sum of these two terms under the relevant constraints. In other words, we have:

$$V(y_t, t) = \max_{x_t} \left[F(y_t, x_t, t) + V(y_t, t+1) \right] \tag{4.15}$$

under the constraints (4.1) and (4.2) for the specific t.

This method of intertemporal optimization, as a succession of static optimization problems, was proposed by Richard Bellman and is called *dynamic programming*. The idea that whatever the choice in period t, the choices for the subproblem that begins in period $t+1$ should be optimal, is known as Bellman's *principle of optimality*. The optimal value function $V(y_t, t)$ is called the Bellman's *value function* and equation (4.16) the *Bellman equation*.

The Bellman equation gives us a recursive method for solving the original optimization problem. The idea is to start from the end and go backward. In period T, there is no future, only the requirement for a given final stock Y_{T+1}. Therefore:

$$V(y_T, T) = \max_{x_T} F(Y_t, x_T, T)$$

under the constraints:

$$\begin{aligned} Y_{T+1} &= Y_T + Q(y_T, x_T, T) \\ G(y_T, x_T, T) &\leq 0 \end{aligned}$$

This is a simple probem of static optimization, which gives us the optimal value function $V(Y_T, T)$. This function can in turn be used in the right-hand side of (4.16) for $t = T - 1$. This equation is then another static problem, which gives us the optimal value function

$V(y_{T-1}, T-1)$. We can continue iun this way until we reach period 0. In pratice, this process provides results for the simplest problems. Analytical solutions exist when the functions F, G, and Q have a very simple form. Where analytical solutions do not exist, we can use numerical solutions, acknowledging that, for many economic applications, we have a better methods than the recursive method to find or characterize the solution.

Note that in the presence of uncertainty, the Bellman equation takes the form:

$$V(y_t, t) = \max_{x_t} \left[F(y_t, x_t, t) + E_t V(y_t, t+1)\right] \tag{4.16}$$

where E is the mathematical expectations operator. To find the Bellamn equations in continuous time, note that from (4.16) takes the form:

$$V(y(t), t) = \max_{x(t)} \left[F(y(t), x(t), t)\Delta t + V(y(t + \Delta t y, t + \Delta t)\right]$$

where Δt is a small time interval. using a Taylor expansion of the last right-hand side term of the equation above, we get:

$$V(y(t + \Delta t y, t + \Delta t) = V(y(t), t) + V_y(y(t), t)\left[y(t + \Delta t) - y(t)\right] + V_t(y(t), t)\Delta t$$

and where: $y(t + \Delta t) - y(t) = Q(y(t), x(t), t)\Delta t$ from (4.1). Plug this expression and the previous to obtain:

$$V(y(t), t) = \max_{x(t)} \left[F(y(t), x(t), t)\Delta t + V(y(t), t) + V_y(y(t), t)Q(y(t), x(t), t)\Delta t + V_t(y(t), t)\Delta t\right]$$

Divide both side by Δt and cancelling $V(y(t), t)$ results in:

$$0 = \max_{x(t)} \left[F(y(t), x(t), t) + V_y(y(t), t)Q(y(t), x(t), t) + V_t(y(t), t)\right]$$

Which is a Bellman Equation in continuous time.

4.4 Present and Current Value Problems

Before going through additional details, we need to precise two fundamental points:

1. Dynamic economic problems often discount future values of the objective function
2. if $T = +\infty$, there is no guarantee that the objective function (4.3) (or the continuous time version (4.10) is converging. Discounting the objective function is one way to guarantee such convergence.

When discounted, the objective functions write:

$$\begin{aligned} \sum_{t=0}^{T} \beta^t F(y_t, x_t, t), & \quad \text{in discrete time} \\ \int_{t=0}^{T} \exp^{-\rho} F(y_t, x_t, t) & \quad \text{in continuous time} \end{aligned} \tag{4.17}$$

with $\rho > 0$ the **rate of time preference** and $\beta = \frac{1}{1+\rho} \in (0, 1)$, the **discount factor**.[3] This two parameters defines the degree of impatience of the economic agents, or how much they subjectively value the present versus the future. In particular, a low β (and a high ρ) means that the agents is relatively impatient (she puts less weight on future value of the objective function).

From there, there are two different (although equivalent) way to frame the problem. Either build the **present-value Hamiltonian** or the **current-value Hamiltonian**. In continuous time, the present-value Hamiltonian is:[4]

$$\mathcal{H} = F(y(t), x(t), t) \exp^{-\rho t} + \tilde{\lambda}(t) Q(y(t), x(t), t)$$

The first-order conditions do not merely change in comparison to Section 4.2.2:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{H}}}{\partial x(t)} = 0: & \qquad F_x \exp^{-\rho t} - \tilde{\lambda}(t) Q_x = 0 \\ \frac{\partial \tilde{\mathcal{H}}}{\partial y(t)} = -\dot{\tilde{\lambda}}: & \qquad F_y \exp^{-\rho t} - \tilde{\lambda}(t) Q_y = -\dot{\tilde{\lambda}}(t) \\ \frac{\partial \tilde{\mathcal{H}}}{\partial \lambda(t)} = \dot{y}(t): & \qquad Q(y(t), x(t), t) = \dot{y}(t) \end{aligned} \tag{4.18}$$

Using an appropriate change of variable, the current-value Hamiltonian is:

$$\mathcal{H} = \tilde{\mathcal{H}} \exp^{\rho t} = F(y(t), x(t), t) + \lambda(t) Q(y(t), x(t), t)$$

and where $\lambda(t) = \tilde{\lambda}(t) \exp^{-\rho t}$. Noting that $-\dot{\tilde{\lambda}}(t) = -(\dot{\lambda}(t) - \rho\lambda(t))$. The FOC becomes:

[3] These two parameters are related as follows: $\beta^t = e^{t \ln \beta} = e^{\left(\ln \frac{1}{1+\rho}\right) t} = e^{[\ln 1 - \ln(1+\rho)]t} = e^{[0 - \ln(1+\rho)]t} = e^{-\rho t}$

[4] Without loss of generality, we exclude the static constraint $G(y(t), x(t), t)$.

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial x(t)} = 0: \quad & F_x - \lambda(t) Q_x = 0 \\
\frac{\partial \mathcal{H}}{\partial y(t)} = -\dot{\lambda}(t) - \rho\lambda(t) \equiv -\dot{\tilde{\lambda}} \exp^{-\rho t}: \quad & F_y - \lambda(t) Q_y - \rho\lambda(t) = -\dot{\lambda}(t) \\
\frac{\partial \mathcal{H}}{\partial \lambda(t)} = \dot{y}(t): \quad & Q(y(t), x(t), t) = \dot{y}(t)
\end{aligned}
\tag{4.19}
$$

Note that using the two first equations in (4.18) (in particular by taking the time derivative of the first equation to eliminate the terms in $\tilde{\lambda}$) leads to the same expression for the second equation of (4.19). Hence, starting with the current-value Hamiltonian may save some computations.

A similar approach can be done in the discrete time problem. The present-value Lagrangian writes:

$$
\tilde{\mathcal{L}} = \sum_{t=0}^{T} \beta^t F(y_, x_t, t) + \tilde{\lambda}_{t+1}[Q(y_t, x_t, t) + y_t - y_{t+1}]
$$

with FOC:

$$
\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial x_t} = 0: \quad & \beta^t F_x + \tilde{\lambda}_{t+1} Q_x = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial y_t} = 0: \quad & \beta^t F_y + \tilde{\lambda}_{t+1} Q_y + \tilde{\lambda}_{t+1} - \tilde{\lambda}_t = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{\lambda}_{t+1}} = 0: \quad & y_{t+1} - y_t = Q(y_t, x_t, t)
\end{aligned}
\tag{4.20}
$$

Using the change of variable $\tilde{\lambda} = \beta^t \lambda_t$, the current-value Lagrangian is now:

$$
\tilde{\mathcal{L}} = \sum_{t=0}^{T} \beta^t F(y_, x_t, t) + \lambda_{t+1}[Q(y_t, x_t, t) + y_t - y_{t+1}]
$$

with the following FOCs:

$$
\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial x_t} = 0: \quad & F_x + \lambda_{t+1} Q_x = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial y_t} = 0: \quad & F_y + \lambda_{t+1} Q_y + \lambda_{t+1} - \beta^{-1}\lambda_t = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_{t+1}} = 0: \quad & y_{t+1} - y_t = Q(y_t, x_t, t)
\end{aligned}
\tag{4.21}
$$

The main differences with the present-value FOCS are that the β^t in the first equation eliminates themselves while in the second equation, β^{t+1} factorizes the term in $t+1$ and the β^t factorizes the term in t, cancelling each other up to one factor β. As in the continuous time case, the two formulation are equivalent once solved, although the current-value approach saves additional computations.

Let us look at discounted problem with the dynamic programming approach. Define:

$$V(y_t, t) = \sum_{t=0}^{T} \beta^t F(y_t, x_t, t) \tag{4.22}$$

subject to:

$$y_{t+1} - y_t = Q(y_t, x_t, t)$$

Note that the constraint implies $x_t \in H(y_t, y_{t+1}, t)$. Problem (4.22) writes:[5]

$$\begin{aligned} V(y_t, t) = \max_{x_t \in H(y_t, y_{t+1})} \sum_{t=0}^{T} \beta^t F(y_t, x_t, t) &= \max_{x_t \in H(y_t, y_{t+1})} \left[F(y_t, x_t, t) + \beta \sum_{t=0}^{T} \beta^t F(y_{t+1}, x_{t+1}, t+1) \right] \\ &= \max_{x_t \in H(y_t, y_{t+1})} \left[F(y_t, x_t, t) + \beta V(y_{t+1}, t+1) \right] \end{aligned} \tag{4.23}$$

To maximize the Bellman equation, write the static lagrangian:

$$\mathcal{L} = F(y_t, x_t, t) + \beta V(y_{t+1}, t+1) + \lambda \left[Q(y_t, x_t, t) + y_t - y_{t+1} \right]$$

The first-order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_t} = 0 : &\quad F_x + \lambda Q_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y_{t+1}} = 0 : &\quad -\lambda + \beta V_y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 : &\quad y_{t+1} - y_t = Q(y_t, x_t, t) \end{aligned}$$

Which gives the system to be solved recursively.

[5] To obtain the different equalities, just decompose the sum with the terms in period t and the remaining terms and recognizing that: $\sum_{t=1}^{T} \beta^t F(y_t, x_t, t) = \sum_{t=0}^{T} \beta^{t+1} F(y_{t+1}, x_{t+1}, t+1) = \beta \sum_{t=1}^{T} \beta^t F(y_t, x_t, t) = \beta V(y_{t+1}, t+1)$.

The Benveniste-Scheinkman condition gives using (4.23):

$$\frac{\partial V(y_t, t)}{\partial y_t} = F_y + \lambda \left[Q_y + 1\right] \Rightarrow \frac{\partial V(y_{t+1}, t+1)}{\partial y_{t+1}} = F_y + \lambda \left[Q_y + 1\right]$$

which can be plugged in the FOCs above.

In continuous time, the discounted Bellman Equation is:[6]

$$-\dot{V}(y(t), t) + \rho V(y(t), t) = \max_{x(t) \in H(y(t),t)} \left[F(y(t), x(t), t) + v_y(y(t), t) Q(y(t), x(t), t)\right]$$

4.5 Transversality Conditions

We have mentioned that initial conditions are usually given a dynamic economic problem, notably because this is part of history. However, we have not deal with terminal conditions, also called **transversality conditions**.

In the case of finite horizon problem where $T < +\infty$, the terminal condition implies that the stock variables at period $T+1$ are nil. This means that the Lagrangian has additional constraints of the form:

$$\omega_{T+1} y_{T+1}$$

whose associated FOCs are:[7]

$$\frac{\mathcal{L}}{\partial y_{T+1}} = 0 : -\beta^T \lambda_{T+1} + \omega_{t+1} \beta^{T+1} = 0$$

Together with a complementary slackness condition:

$$\begin{aligned} \frac{\mathcal{L}}{\partial \beta^{T+1} \omega_{T+1}} &= y_{t+1} \geq 0 \\ \beta^{T+1} \omega_{T+1} \frac{\mathcal{L}}{\partial \beta^{T+1} \omega_{T+1}} &= \beta^T \lambda_{T+1} y_{t+1} = 0 \end{aligned}$$

Since λ_{T+1} is not nil (because this would imply an infinite marginal benefits in the objective function), this means that $y_{t+1} = 0$.[8]

In continuous time, the equivalent condition at point in time T is:

[6]The sketch of the proof being more complex here, it is left as an exercise.

[7]remember that y_{T+1} comes also in the dynamic constraint involving Q.

[8]For instance, in the example of a household with wealth y and consumption level x, this means that at the final period T, the household consume all its wealth since it has no use for the periods after.

$$\lambda(T)y(T) = 0$$

But what happens when we are looking at an infinite horizon problem ? The latter saves a lot of complication (basically, everything is equivalent as a two-period problem with "today" and "tomorrow" carried over and over) and is often assumed in dynamic problems. Then, we need to prevent the agents to shift indefinitely its decisions on the stocks y over and over.Hence, the equivalent transversality condition in discrete and continuous time in infinite horizon problems are:

$$\lim_{t \to +\infty} \lambda_t y_t = 0 \quad \text{in discrete time}$$
$$\lim_{t \to +\infty} \lambda(t) y(t) = 0 \quad \text{in continuous time}$$

> **⚠ Warning**
>
> This is an abusive interpretation as Kamihigashi (2008) highlights. Transversality conditions and no-Ponzi conditions, which is what we just described, have different roles, in particular transversality conditions are **necessary** in any infinite horizon problem. Be sure to check that there are satisfied (In the next chapters, they are.)

4.6 General Procedures

To summarize and give a "cookbook" approach, we present a general strategy to solve dynamic optimization problem.

For the sequence problem:

1. Write the Lagrangian/Hamiltonian
2. Find the first-order conditions
3. Obtain the difference/differentials equations in control and state variables
4. Use the techniques in Chapter 2 or in Chapter 3 to characterize the solution.

For the recursive approach, using the Bellman equation, things may differ as you can return to a sequence problem using the Benveniste-Scheinkman (and thus start at the third bullet-point of the previous list after defining the Bellman equation and find the first-order conditions). Otherwise You may also use a "guess and verify approach", either analytically or numerically, of the value function or the policy function to obtain the solution.

! Second-Order Conditions

Although we do not discuss second-order conditions in this chapter, one needs to check if the Lagrangian/Hamiltonian or the Bellman equation is concave (respectively convex) in order to maximize (respectively minimize) the original problem.

4.7 Applications

To be completed(you can refer to the next part though). One can check Dixit's manual on optimization.

Part II

Benchmark Models in Growth and Fluctuations Theory

5 The Solow Model

5.1 Introduction

The Solow model constitutes a cornerstone in the growth analysis. Pre-solovian Growth Theory, turning mainly around Harrod and Domar contributions, concluded on the instability of long-run economic growth. Such conclusions were at odds at observed data. Although the Solow model has many pitfalls, it constitute a good starting point to discuss the structure of the many macroeconomic models that usually keep the same production structure.

The contributions of Solow takes place within a range of observations, called Kaldor's stylized facts (after Kaldor, 1963). Namely:

1. GDP per capital grows over time
2. Per capital capital stock grows over time
3. Capital interest rate is roughly constant over long period
4. the capital-to-GDP ratio is roughly constant
5. Capital income and labor income share are roughly constant over long period
6. GDP per capita growth rates are different between countries

Note a consequence of point 5. is that, because the labor income share wL/Y is constant and that GDP per capita Y/L grows, the wage rate w has necessarily the same growth rate of the GDP per capita.

These stylized fact are the benchmark points to evaluate any model of economic growth. [1]

In the rest of this chapter, we first discuss an important property of benchmark growth model. Afterwards, we describe and solve the Solow model in discrete time (the continuous time being studied in Appendix). We study the equilibrium growth rates and transitional dynamics and also discuss further topics such as the properties of factor prices, the Golden Rule of capital accumulation and growth accounting.

[1] Note that some of these facts have evolved in recent years, in particular the stability of income shares. Furthermore, additional facts can be included. See the excellent chapter of Jones (2005) in the Handbook of Economic Growth.

5.2 A general property: the balanced-growth path

Before starting with the presentation of the Solow model, we derive an important and useful property of any one-sector growth model provided it includes capital accumulation. This property, called *the balanced growth path* (or BGP), focuses on the equilibrium growth rates of aggregate quantities. We consider a simple economy characterized (for the moment) by two equations:

$$K_{t+1} = I_t + (1-\delta)K_t \tag{5.1}$$

$$Y_t = C_t + I_t \tag{5.2}$$

Equation (5.1) represents the capital accumulation equation[2] while Equation (5.2) is the aggregate resource constraint of the economy. Variables K_t, I_t, C_t and Y_t the aggregate stock of capital installed at date t and investment realized at date t, aggregate consumption at date t and GDP at date t respectively. The parameter $\delta \in (0,1)$ represents depreciation of the capital stock at each period.

Definition 5.1. A balanced-growth path is a path $\{Y_t, C_t, I_t, K_t\}_{t\geq 0}$ along which the quantities Y_t, K_t, and C_t are positive and growth at constant rates, which we denotes g_Y, g_K and g_C respectively.

We can show then:

Proposition 5.1. *Let $\{Y_t, C_t, I_t, K_t\}_{t\geq 0}$ be a balanced-growth path. Then, given the capital accumulation equation and the aggregate resource constraint, the following holds:*

1. *If there is a BGP, then $g_Y = g_K = g_C = g_I$ and the ratios K/Y, C/Y and I/Y are constant.*
2. *if K/Y and C/Y are constant, then Y, K, C and I all grow at the same constant rate, i.e. there is a balanced-growth path with the additional property that all aggregate variables grow at the same growth rate.*

[2] In continuous time, this equation writes: $\dot{K}(t) = I(t) - \delta K(t)$.

Proof

For the first part of the proposition: by definition, g_Y, g_C, g_I and g_K are constant. Rewriting Equation (5.1):

$$\frac{\Delta K_{t+1}}{K_t} = g_K = \frac{I_t}{K_t} - \delta \Leftrightarrow \frac{I_t}{K_t} = g_K + \delta$$

Since the right-hand side of this equation is constant, then the ratio I_t/K_t is constant and therefore I_t and K_t have the same growth rate ($g_K = g_I$). We can focus on the aggregate resource constraint:

$$Y_t = C_t + I_t \Rightarrow \Delta Y_t = \Delta C_t + \Delta I_t \Rightarrow g_Y = \frac{\Delta Y_t}{Y_t} = \frac{C_t}{Y_t} g_C + \frac{I_t}{Y_t} g_I$$

Since $\frac{I_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ and $g_I = g_K$, this equation becomes:

$$g_Y = \frac{C_t}{Y_t}(g_C - g_K) + g_K$$

There are two possibilities: either $g_C = g_K$, from which we directly derive that $g_Y = g_K = g_C = g_I$ or $g_K \neq g_C$ and the last equation rewrites:

$$\frac{C_t}{Y_t} = \frac{g_Y - g_K}{g_C - g_K}$$

The right-hand side is constant, meaning that the ratio of C_t over Y_t is also constant, and therefore they grow at the same rate. Then, the RHS is equal to unity and therefore $C_t = Y_t$ which contradicts the fact $I_t > 0$. Hence, $g_C \neq g_K$ is not possible according to the definition of a BGP and implies that $g_C = g_K = g_Y = g_I$.
To prove the second part of the proposition: Assume K/Y and C/Y are constant. Then, by definition, $g_Y = g_K = g_C$ and therefore $\frac{I_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ meaning $g_I = g_K$. To show that the growth rate is constant, we use Equation (5.1):

$$g_k = \frac{I_t}{K_t} - \delta$$

where the right-hand side is constant.

This is a quite important result as it implies that there can't be persistent deviations of aggregate quantities growth rate. Furthermore, this not only gives us a starting point to link theoretical quantities with observable but also regularities to reproduce or not given the structure we give to the model.

5.3 Assumptions

On the demand side, we consider that households are identical and follow a linear keynesian consumption function. In particular, we assume the aggregate consumption is $C_t = (1-s)Y_t$ with the parameter $s \in (0,1)$, the marginal propension to save. It follows that aggregate savings is given by: $S_t = sY_t$. We also assume that households supply a fixed amount of labor.

Population N_t grows at a given and fixed rate n (so that labor supply continuously grows). The good Y can be either consumed or transformed into capital stock through investment. The production function $Y_t = F(K_t, A_t Lt)$ is homogeneous of degree 1 (i.e. linear homogeneity), satisfies Inada conditions, is increasing and concave in its arguments. L_t is labor used to produced Y_t while A_t represents exogenous technical progress and grows at an exogenous rate g.[3] Without loss of generality,[^we could indeed provide a full, but more complex, analysis just using the property given above.] we can assume a Cobb-Douglas production function $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$ where $\alpha \in (0,1)$ is the output elasticity of capital (i.e. the percentage increase in output after a one-percent change in capital stock). Because of linear homogeneity, α is also the capital income share (or the percentage capital income earned in the production process).

! Important

Altough A_t is called "technical (or technological) progress", keep in mind that this is a catch-up variable and does not capture only the technological level of a country. Indeed, a more agnostic interpretation would also include political stability, quality of institutions such as laws, patents and so on.

The economy is competitive (there is no market power or externalities). The model focus on a closed economy (no exporations or importations). There is no government intervention and the economy is at full employment such that $L_t = N_t$. These assumptions are inoffensive to the main conclusions we'll get thereafter.

5.4 Solving the model

The good market implies that $S_t = I_t$ at each date. Hence, we may write:

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{5.3}$$

[3] Introducing technical progress as we did is called "Harrod-neutral". The two other for of technological progress are "Hick-Neutral" ($Y_t = A_t F(K_t, L_t)$) and "Solow-Neutral" ($Y_t = F(A_t K_t, L_t)$) but are not compatible with balanced-growth.

Since $C_t = Y_t - I_t = (1-s)Y_t$ and $Y_t = F(K_t, A_tL_t)$ with A_t and L_t evolving exogenously, it is enough to use Equation (5.3) to describe the whole trajectory of the economy, although we cannot provide a solution of this equation. To do so, we need to rewrite ([-#eq-aggregate-equilibrium]) in effective labor unit. Denote GDP in effective labor $\tilde{y}_t = \frac{Y_t}{A_tL_t}$ and effective labor capital stock $\tilde{k}_t = \frac{K_t}{A_tL_t}$, then we can derive:

$$\frac{K_{t+1}}{A_tL_t} = \frac{sY_t + (1-\delta)K_t}{A_tL_t} = s\tilde{y}_t + (1-\delta)\tilde{k}_t$$

Using the assumption on the evolution of population and technical progress (e.g. $L_{t+1} = (1+n)L_t$ and $A_{t+1} = (1+g)A_t$):

$$\frac{A_{t+1}L_{t+1}}{A_tL_t}\frac{K_{t+1}}{A_{t+1}L_{t+1}} = (1+n)(1+g)\tilde{k}_{t+1} = s\tilde{y}_t + (1-\delta)\tilde{k}_t$$

To get rid of $\tilde{y}_t$, we use the production function:

$$\frac{Y_t}{A_tL_t} = \tilde{y}_t = \frac{K_t^\alpha (A_tL_t)^{1-\alpha}}{A_tL_t} = \left(\frac{K_t}{A_tL_t}\right)^\alpha \left(\frac{A_tL_t}{A_tL_t}\right)^{1-\alpha} = \tilde{k}_t^\alpha$$

The evolution of the capital stock in terms of effective labor is then:

$$(1+n)(1+g)\tilde{k}_{t+1} = s\tilde{k}_t^\alpha + (1-\delta)\tilde{k}_t \tag{5.4}$$

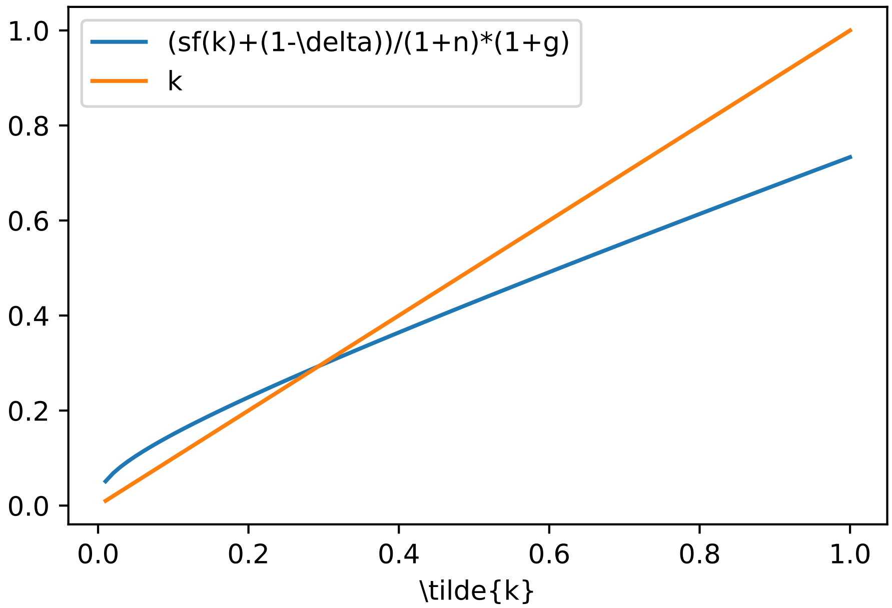
Equation (5.4) is the "fundemental Solow equation", which is a non-linear first-order difference equation in $\tilde{k}_t$. It displays two difference forces in the evolution of this variable. On the first hand, capital stock in terms of effective labor increases as households save (and therefore invest) more, that is the higher $s\tilde{k}_t^\alpha$. On the other hand, at each period, the capital stock in terms of effective labor depletes because of physical depreciation (effect of δ) but also because there are more people (L_t increasing at rate n) and that workers are more efficient (A_t increasing at rate g). [4] To say it differently, as there are more productive $\frac{K_{t+1}}{A_{t+1}L_{t+1}}$ decreases. Hence, the economy requires to invest more to keep $\tilde{k}_{t+1}$ higher.

We derive the steady state by putting $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$ and solving Equation (5.4):[5]

[4] This can be seen as the RHS can be divided by $(1+n)(1+g)$.

[5] Note that when developing $(1+n)(1+g)$, we considered that $ng \approx 0$ because both are close to 0 enough.

$$
\begin{aligned}
&(1+n)(1+g)\tilde{k}^* = s\left(\tilde{k}^*\right)^\alpha + (1-\delta)\tilde{k}^* \\
&\Rightarrow (\delta+n+g+ng)\tilde{k}^* = s\left(\tilde{k}^*\right)^\alpha \\
&\Rightarrow \tilde{k}^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$



Note that the steady state stock of capital in terms of effective labor is increasing with respect to the marginal propension to save s and is decreasing with respect to the depreciation rate, the population growth rate n and the technical progress growth rate g. The remaining steady state variable of interest are:

$$
\tilde{y}^* = \left(\tilde{k}^*\right)^\alpha = \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}, \quad \tilde{c}^* = (1-s)\left(\tilde{k}^*\right)^\alpha = \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}
$$

Note that taking first-order approximation of Equation (5.4) in the neighborhood of the steady state implies that $|\frac{d\tilde{k}_{t+1}}{d\tilde{k}_t}| < 1$ and provides stability.

5.5 Growth rates

It is important to understand that by definition the growth rates of $\tilde{k}_t$, $\tilde{y}_t$ and $\tilde{c}_t$ are zero when the economy reaches the steady state[6]. However, by definition of the variables in term of effective labor, we have:

$$\begin{aligned} K_t = A_t L_t \tilde{k}_t, \quad & Y_t = A_t L_t \tilde{y}_t, \quad & C_t = A_t L_t \tilde{c}_t \\ k_t = A_t \tilde{k}_t, \quad & y_t = A_t \tilde{y}_t, \quad & c_t = A_t \tilde{c}_t \end{aligned}$$

Where k_t, y_t and c_t are per capita capital stock, per capita GDP and per capita consumption.

Hence, as $\tilde{k} \to \tilde{k}^*$, aggregate variable K_t, Y_t and C_t grow at rate $n+g$ while per capita variables k_t, y_t and c_t grow at rate g. Indeed for the aggregate GDP:[7]

$$\begin{aligned} & ln(Y_{t+1}) = ln(A_{t+1}) + ln(L_{t+1}) + ln(\tilde{y}^*) \\ & \Rightarrow ln(Y_{t+1}) - ln(Y_t) \equiv ln(1 + g_Y) = ln(A_{t+1}) - ln(A_t) + ln(L_{t+1}) - ln(L_t) \\ & \Rightarrow g_y = n + g \end{aligned}$$

Similarly for the GDP per capita:

$$ln(y_{t+1}) - ln(y_t) = g$$

It is easily verified (and automatically satisfy by the properties of the BGP) that K_t and C_t also grow at rate $n+g$ and k_t and c_t grow at rate g.

Now we can understand why the conclusions of Solow are important. As economists studying growth, we are interested into growth in per capita quantities. Assuming away technical progress (by putting $g = 0$) implies that the per capita variable reach the steady state and stop to grow, which is at odd with what is observed in the data. Another way to say it, growth through an extensive process (increasing private inputs such as capital stock and labor, as hours of work or number of people working) cannot be a source of persistent growth. This is a direct consequence of marginal decreasing returns of capital: even in the favorable situation of linear homogeneity of the production (e.g. doubling all inputs doubles output), GDP per capita has decreasing returns in terms of stock of capital per capita meaning that any increase in k_t makes GDP larger but less and less. To explain persistent growth, this requires an additional variables which is exogenous in the model and takes the form of the catch-up variable A_t, the so-called technological progress.

[6] This can be seen by dividing Equation (5.4) by $\tilde{k}_t$ on both sides and noting that $\frac{\tilde{k}_{t+1}}{\tilde{k}_t} = (1 + g_{\tilde{k}})$.

[7] Note that for any variable x_t that grows at variable g_x such that $x_t = (1 + g_x)x_{t-1} \Rightarrow ln(x_t) - ln(x_{t-1}) = ln(1 + g_x) \approx g_x$. We could also directly take the ratio $Y_{t+1}/Y_t = 1 + g_Y = (1 + n)(1 + g) \approx 1 + n + g$.

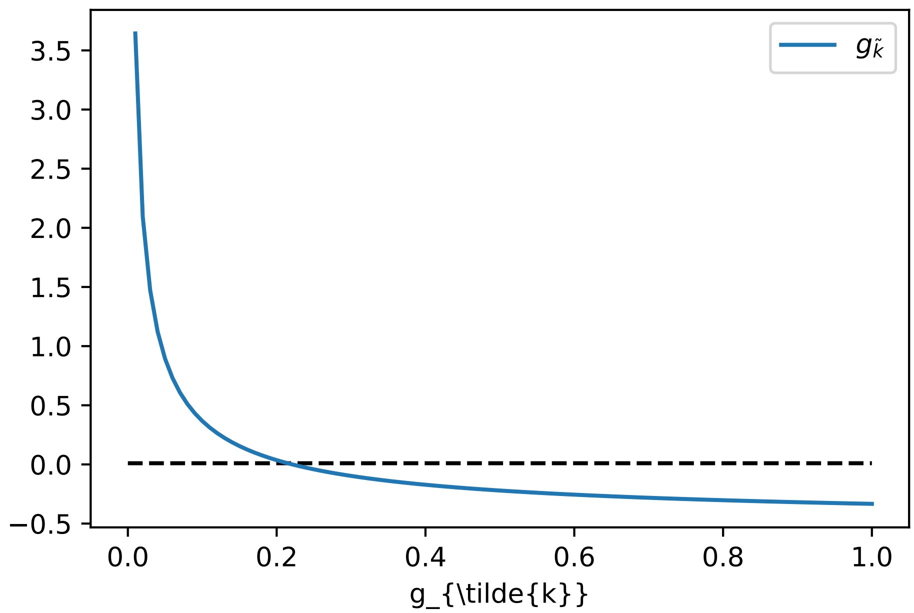
5.6 Transitional dynamics

We can use Equation (5.4) to study growth rate outside of the steady state. Rewrite it such that:

$$\Delta\tilde{k}_{t+1} \equiv \tilde{k}_{\mathrm{t+1}} - \tilde{k}_{\mathrm{t}} = \frac{s\tilde{k}_t + (1-\delta)\tilde{k}_{\mathrm{t}} - (1+n+g+ng)\tilde{k}_{\mathrm{t}}}{(1+n)(1+g)}$$

$$= \frac{sf\left(\tilde{k}_{\mathrm{t}}\right) - (\delta+n+g+ng)\tilde{k}_{\mathrm{t}}}{(1+n)(1+g)} \simeq s\tilde{k}^{\alpha} - (\delta+n+g)\tilde{k}_{\mathrm{t}}$$

$$\Rightarrow g_{\tilde{k}} \equiv \frac{\Delta\tilde{k}_{\mathrm{t+1}}}{\tilde{k}_{\mathrm{t}}} \simeq s\tilde{k}_t^{\alpha-1} - (\delta+n+g)$$



Since the first term on the right hand side is decreasing in $\tilde{k}_t$, for any $\tilde{k}_t < \tilde{k}^*$, the growth rate of $\tilde{k}$, $g_{\tilde{k}}$ is positive and decreases (and conversely for any $\tilde{k}_t > \tilde{k}^*$).

5.7 Behavior of the interest rate and the wage rate

Let us look at the firms' profit maximization problem:

$$\pi = \max_{K(t),L(t)} \{K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - w(t)L(t) - (r(t)+\delta)K(t)$$

whose first-order conditions are:

$$
\begin{aligned}
\alpha K^{\alpha-1}\left(A(t)L(t)\right)^{1-\alpha} = \alpha\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha-1} &\equiv \alpha\tilde{k}(t)^{\alpha-1} = r(t)+\delta \\
(1-\alpha)K^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} = (1-\alpha)A(t)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} &\equiv (1-\alpha)A(t)\tilde{k}(t)^{\alpha} = w(t)
\end{aligned}
\tag{5.5}
$$

Equations in (7.1) show two different properties for factor prices. In particular, the interest rate depends only on the effective capital stock while the wage rate depends on both effective capital stock and (proportionaly) technical progress. As a result, when the economy reaches the steady state, the interest rate becomes constant while the wage rate grows at the same rate than the equilibrium growth rate of per capital GDP, capital stock and consumption.

5.8 The Golden Rule

We have seen that the steady state effective capital stock and consumption are given by:

$$
k^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}, \quad c^* = (1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}
$$

For given parameters others than s, we can note that the saving rate as contradictory effect on c^*: one the first hand, a decrease in s increase the share of consumption given by $(1-s)$ but it also reduces the effective capital stock at the steady state, and therefore the amount produced. [8]

The question is then: does it exist a saving rate that maximizes steady state consumption ? Regarding the previous paragraph, the answer is a definitive "yes" as shown below. Define $c(s) = c^*(s)$ and $k(s) = k^*(s)$ (that is recognize that steady state consumption and capital stock depends on the saving rate), then:

$$
c^{GR} : \max_{s}\{c(s) = k(s)^{\alpha} - (\delta+n+g)k(s)\}
$$

which gives:

$$
\alpha k(s)^{\alpha-1}\frac{\partial k(s)}{\partial s} = (\delta+n+g)\frac{\partial k(s)}{\partial s} \Rightarrow k^{GR} = \left(\frac{\alpha}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}
$$

[8] Note that this is independent of considering population growth or growth technical progress.

the Golden rule states that maximum steady state of consumption is obtained if the marginal product of capital is equal to the economic rate of depletion of capital stock $\delta+n+g$. [9] One can show that the golden rule is generically not satisfied, unless $s = \alpha$. If $\alpha < s$ (respectively $s < \alpha$) then, the economy over-accumulates capital stock ($k^* > k^{GR}$) (respectively under-accumulate capital stock, $k^* < k^{GR}$).

Satisfying or not the golden rule is important as it has consequences on future cohorts of unborn yet people (and therefore intergenerational equity issues): too low savings means that current living households should increase the rate at which they are saving and accumulating capital while too high savings means we can at the same time increase consumption and capital accumulation. While the former characterizes under-accumulation of capital , the latter represents over-accumulation of capital. This can be summarized by the concept of dynamic efficiency.

Definition 5.2. Dynamic efficiency is a situation where an economy under-accumulates capital stock, that is $k^* < k^{GR}$ while a dynamic inefficient economy over-accumulates capital stock, that is $k^* > k^{GR}$.

5.9 Growth Accounting

Having achieved the analysis, we can use the structure of the model to measure the contributions of each factor to growth of the GDP per capita.

From the production function $Y = K^\alpha (AL)^{1-\alpha}$, take logs and differentiate to get it in terms of growth rates:

$$g_Y = \alpha g_K + (1-\alpha)(n+g) g_Y - n = g_y = \alpha(g_K - n) + (1-\alpha) g$$

Note that we can easily find data describing growth rates of GDP per capita g_y, capital stock g_K and demographic growth n. Furthermore, α is the capital income share and is around $\alpha = 0.3$. Hence, we can solve this relationship for g and plug in values:

$$g = \frac{1}{1-\alpha} g_y - \frac{\alpha}{1-\alpha}(g_K - n)$$

Once you obtain estimated values of g, let's write $\hat{g}$, then the contribution to growth of GDP per capita can be computed:

$$g_y = \alpha(g_K - n) + (1-\alpha)\hat{g}$$

[9] In a more general setup, without specifying the production function $f(k)$, the golden rule is given by: $f'(k^{GR}) = (\delta + n + g)$

The first term on the RHS is the effect of capital stock per capita and contributes around one third to the growth rate of GDP per capita while the second term is the effect of productivity and contributes up to two-third of GDP per capita growth.

5.10 Appendix

5.10.1 Deriving the general production function for GDP per capita and effective labor GDP

Consider the general production function $Y_t = F(K_t, A_t L_t)$. Since it has homogeneity of degree one, we can write:

$$Y_t = F(A_t L_t \tilde{k}, A_t L_t) = A_t L_t F(\tilde{k}_t, 1) = A_t L_t f(\tilde{k}_t) \Rightarrow \tilde{y}_t = f(\tilde{k}_t)$$

With $f'(\tilde{k}_t) > 0$ and $f''(\tilde{k}_t) < 0$, that is the production function in intensive form is increasing and concave.

Obviously, if we study per capita GDP, you get:

$$\frac{Y_t}{L_t} = y_t = \frac{f(K_t, A_t L_t)}{L_t} = f(\frac{K_t}{L_t}, A_t) = f(k_t, A_t)$$

We can also derive useful expressions for the interest rate and the wage rate:

$$\pi_t = A_t L_t f(\tilde{k}_t) - w_t L_t - r K_t$$

Marginal productivity of capital equations the interest rate:

$$\frac{\partial \pi_t}{\partial K_t} = 0 \Rightarrow A_t L_t f'(\tilde{k}_t) \frac{\partial \tilde{k}_t}{\partial K_t} = r_t \Rightarrow A_t L_t f'(\tilde{k}_t) \frac{1}{A_t L_t} = r_t \Rightarrow f'(\tilde{k}_t) = r_t$$

Marginal productivity of labor equates the wage rate:

$$\begin{aligned}
\frac{\partial \pi_t}{\partial L_t} = 0 &\Rightarrow A_t f(\tilde{k}_t) + A_t L_t f'(\tilde{k}_t) \frac{\partial \tilde{k}_t}{\partial L_t} = w_t \\
&\Rightarrow A_t f(\tilde{k}_t) - A_t L_t f'(\tilde{k}_t) \frac{K_t}{A_t L_t^2} = A_t \left(f(\tilde{k}_t) - f'(\tilde{k}_t) \frac{K_t}{A_t L_t} \right) \\
&\Rightarrow A_t (f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)) = w_t
\end{aligned}$$

5.10.2 The model in continuous-time

The equation for aggregate capital accumulation is now:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{5.6}$$

Demographic changes and technological change are given by:[10]

$$\dot{L}(t) = nL(t), \quad \dot{A}(t) = gA(t)$$

and the production function is given by: $Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha}$.

Households save a fraction $s \in (0, 1)$ of their income.

Using the good market equilibrium condition $I(t) = S(t) \Rightarrow I(t) = sY(t)$ into Equation (5.6), we get:

$$\dot{K}(t) = sY(t) - \delta K(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t)$$

As in the discrete time version, we want to express the evolution of the capital stock in term of effective labor $\dot{\tilde{k}}(t)$:

$$\begin{aligned}
\dot{\tilde{k}}(t) = \dot{\left(\frac{K(t)}{A(t)L(t)}\right)} =& \frac{\dot{K}(t)A(t)L(t) - K(t)(\dot{A}(t)L(t) + A(t)\dot{(L)}(t))}{(A(t)L(t))^2} \\
&= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)} \left(\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right) \\
&= \frac{\dot{K}(t)}{A(t)L(t)} - (n+g)\,\tilde{k}(t) \\
&= \frac{sK(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} - \delta K(t)}{A(t)L(t)} - (n+g)\,\tilde{k}(t) \\
&= s\tilde{k}(t)^{\alpha} - (\delta + n + g)\tilde{k}(t)
\end{aligned}$$

The steady state is obtained by equation $\dot{\tilde{k}}(t) = 0$ and $\tilde{k}(t) = \tilde{k}^*$ and gives:

$$\tilde{k}^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$$

which is equivalent to the discrete time version of the model.

[10] The solution of this exogenous process are respectively $L(t) = e^{nt}$ and $A(t) = e^{gt}$.

The GDP per capita and its growth rate are given by:

$$y(t) = k(t)^{\alpha} A(t)^{1-\alpha} \Rightarrow \frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)} + (1-\alpha) g$$

which implies that $g_y = g_k = g$ using the BGP properties.

6 The Ramsey-Cass-Koopmans Model

6.1 Introduction

The Solow model assumes exogenous and constant savings rate, when the savings are the source of capital accumulation and are a decision variable for the savers (households). The Ramsey model endogenizes consumption and savings decisions.

We will see that in the steady-state the saving rate in the Ramsey model is constant, similar to Solow model. Therefore, basically we will simply re-examine the results of the Solow model, while relaxing the assumption of exogeneity of the savings.

6.2 The Model

A large part of the model is similar to the Solow model, i.e. the production structure. The new element is the many-periods horizon problem of the households. We are going to study the infinite-horizon problem in continuous-time. [1]

- Neoclassical production function: $Y(t) = F(K(t), A(t)L(t))$, where $A(t)$ is the labor augmenting technology
- One-sector model of growth: both capital and consumption goods are produced with the same technology
- A continuum of infinitely lived and identical households of mass $L(t)$
- The representative household is endowed with a unit of labor and chooses its consumption $c(t)$, labor supply (and the evolution of assets $\dot{b}(t)$) to maximize the lifetime utility U, where:^[Since the utility here should be perceived in cardinal sense, the households maximizes simply its utility multiplied by the size of the representative households.}

$$U = \int_0^\infty u(c(t))L(t)e^{-\rho t}dt$$

[1] The discrete-time version of the model is found in Appendix. This is motivated by pedagogical reasons as we have considered the discrete-time Solow model in the main body in the previous chapter.

$u(c(t))$ is the instantaneous utility from consumption of amount $c(t)$ of final good in percapita terms. The instantaneous utility function is increasing and concave in $c(t)$ (i.e., $u' > 0, u'' < 0$) and satisfies the Inada conditions (i.e., $\lim_{c\to 0} u'(c) = \infty, \lim_{c\to\infty} u'(c) = 0$). The concavity implies that households prefers to smooth consumption over time. The pure rate of time preference is $\rho > 0$. The budget constraint written in per capita terms of households is $\dot{b}(t) = (r(t) - n)b(t) + w(t) - c(t)$

We assume two exogenous evolution for population and technology: - Population grows at exogenous rate $\frac{\dot{L}(t)}{L(t)} = n, L(0) > 0$ (taken as given) - Technology grows at exogenous rate $\frac{\dot{A}(t)}{A(t)} = g_A, A(0) > 0$ (taken as given)

6.3 Individual Problems and Market equilibrium

The firm side is similar to Solow model. Formally, setting the final goods as numeraire the representative firm's optimization problem is

$$\max_{K(t),L(t)} \pi(t) = F(K(t), A(t)L(t)) - R(t)K(t) - w(t)L(t)$$

Therefore, the first order conditions (optimal rules) are

$$\begin{aligned}
\frac{\partial \pi(t)}{\partial K(t)} &= 0 \Leftrightarrow F_K = R(t) \\
\frac{\partial \pi(t)}{\partial L(t)} &= 0 \Leftrightarrow F_L = w(t)
\end{aligned}$$

The representative households chooses consumption path to maximize its lifetime utility. Its means of savings is accumulation of capital. Formally, the households' problem is

$$\begin{aligned}
&\max_c U = \int_0^\infty u(c(t)) e^{-(\rho-n)t} dt, \\
&\text{s.t.} \\
&\dot{b}(t) = (r(t) - n)b + w(t) - c(t), \\
&b(0) \text{ given.}
\end{aligned}$$

If written in terms of current value Hamiltonian the households's problem is

$$\max_{c(t),b(t)} H = u(c(t)) + \lambda(t)[(r(t)-n)b(t) + w(t) - c(t)],$$

and

$b(0)$ given,

where the $\lambda(t)$ is the shadow price of a unit of assets. Therefore, the optimal rules are

$$\frac{\partial H}{\partial c(t)} = 0 \Leftrightarrow u'(c(t)) = \lambda(t)$$
$$\dot{\lambda}(t) = \lambda(t)(\rho - n) - \frac{\partial H}{\partial b(t)} = \lambda(t)(\rho - r(t)),$$
$$\lim_{t\to\infty} b(t)\lambda(t)e^{-(r-n)t} = 0$$

From the first optimal rule it follows that

$$\dot{\lambda}(t) = \dot{c(t)}u''(c(t))$$

Therefore,

$$r(t) - \rho = -\frac{\dot{q}(t)}{q(t)} = -\frac{\dot{c}(t)}{c(t)}\frac{c(t)u''(c(t))}{u'(c(t))}$$

or the optimal consumption path is

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{u''(c(t))c(t)}(r-\rho)$$

The transversality conditions states that the value of the current asset holdings in infinity is zero. Formally, this is part of the open boundary problem given by the maximization of H.

Note that $\frac{\dot{c}(t)}{c(t)} > 0$ if $r(t) - \rho > 0$. The sensitivity of the growth of consumption to $r(t) - \rho$ is higher, the lower is $-\frac{u'(c(t))}{u''(c(t))c(t)}$, which is the intertemporal elasticity of substitution (or IES). This elasticity is a measure of the responsiveness of consumption to changes in the marginal utility, i.e., it measures the willingness to deviate from consumption smoothing. In a special case of constant intertemporal elasticity of substitution (CIES) utility function

$$u((t)) = \frac{c(t)^{1-\theta} - 1}{1-\theta}, \quad \theta > 0$$

the growth rate of consumption is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho)$$

since

$$-\frac{u'(c(t))}{u''(c(t))c(t)} = -c(t)^{-\theta-1}\frac{1}{-\theta c(t)^{-\theta-1}} = \frac{1}{\theta}$$

The parameter θ is thereforte the inverse of the intertemporal elasticity of substitution.

The CIES assumption is preserved in what follows.

The equilibrium in the asset market delivers again

$$R(t) = r(t) + \delta$$
$$B(t) = K(t)$$

This gives the law of motion for capital $\dot{K}(t)(t) = Y(t) - C(t) - \delta K$, given that $Y = F(K, AL) = R(t)K(t) + w(t)L(t)$. The last equation is implied by the homogeneity of degree one assumption and states that the final good producer earns zero profit under perfect competition in final good market.

> **Warning**
>
> Note that we could also have written the households' problem where they would own the capital stock. In such case, the interest paid by firms to households would be $r(t)$ and the per capita households budget constraint is $\dot{k}(t) = (r(t) - n - \delta)k(t) + w(t) - c(t)$. This leads to an equivalent results.

6.4 Balanced growth path

All variables of the model need to grow at constant rates (BGP). On a BGP $\frac{\dot{c}(t)}{c(t)} \equiv g_c$ is constant. We have CIES utility function and ρ is a constant parameter, therefore,

$$g_c = \frac{1}{\theta}(r(t) - \rho)$$

On the BGP, therefore the interest rate r should be constant. In our setup, constant interest rate then will imply that savings rate is constant. The intuition behind is that on the BGP there should be no shifts in the shares of aggregates (notice that $C + S = Y$). Use the constant returns to scale assumption and write

$$r-\delta = \frac{\partial F(K, AL)}{\partial K} = \frac{\partial F\left(\frac{K}{AL}, 1\right)}{\partial \frac{K}{AL}} = f'\left(\frac{K}{AL}\right)$$

Given that $f'' < 0$ the ratio $\frac{K}{AL}$ should be constant on the BGP in order to have $r = \text{const.}$ Similarly, given that the ratio $\frac{K}{AL}$ is constant from the constant returns to scale assumption it follows that

$$\frac{Y}{K} = F\left(1, \frac{AL}{K}\right) = \text{ const.}$$

Given that $\frac{Y}{K}$ is constant on the balanced growth path from the law of motion of capital it follows that the ratio $\frac{C}{K}$ also should be constant,

$$\frac{C}{K} = \frac{Y}{K} - \delta - g_K$$

Therefore, $g_K = g_Y = g_C$. Moreover, from $\frac{Y}{K} = F\left(1, \frac{AL}{K}\right)$ it follows that on the balanced growth path $g_K = g_Y = g_C = n + g_A$.

In order to derive the steady-state and to characterize the transition dynamics, redefine the model in units of effective labour, i.e., AL. Let $\tilde{y} \equiv \frac{Y}{AL}, \tilde{k} \equiv \frac{K}{AL}$, and $\tilde{c} \equiv \frac{C}{AL}$. Also, for $Y = AL \times F\left(\frac{K}{AL}, 1\right) \equiv AL \times f(\tilde{k})$.

From these definitions it follows that $R = F_K = f'(\tilde{k}), w = F_L = f(\tilde{k}) - f'(\tilde{k})\tilde{k}$. In the steady-state $g_{\tilde{y}} = g_{\tilde{k}} = g_{\tilde{c}} = 0$. The steady-state and the transition dynamics of the model can be summarized by the following system of equations

$$\begin{aligned}
&\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta}\left[f'(\tilde{k}(t)) - \delta - \rho - \theta g_A\right] \\
&\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{f(\tilde{k}(t))}{\tilde{k}(t)} - \frac{\tilde{c}(t)}{\tilde{k}(t)} - (\delta + n + g_A) \\
&\tilde{k}(0) \text{ given} \\
&\lim_{t\to\infty} \tilde{k}(t)\lambda(t)e^{-(r-n-g_A)t} = 0
\end{aligned} \tag{6.1}$$

The first equation follows from the optimal path of consumption given that

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{c}(t)}{c(t)} - g_A$$

and

$$r = f'(\tilde{k}(t)) - \delta$$

The second equation follows from the law of motion of capital given that

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{k}(t)}{k(t)} - g_A = \frac{\dot{K}(t)}{K(t)} - (n + g_A)$$

In the steady-state $\frac{\dot{\tilde{\tilde{k}}}(t)}{\tilde{\tilde{k}}(t)} = \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = 0$, such that we can solve for the steady-state values of $\tilde{c}$ and $\tilde{k}$ from (7.2). Let $F(K(t), A(t)L(t)) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$ and denote any stationary variable x by x^*:

$$\begin{aligned}
&f'\left(\tilde{k}^*\right) = \delta + \rho + \theta g_A \\
\Rightarrow &\tilde{k}^* = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{1}{1-\alpha}} \Rightarrow f\left(\tilde{k}^*\right) = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{\alpha}{1-\alpha}}, \\
&\tilde{c}^* = f\left(\tilde{k}^*\right) - (\delta + n + g_A)\,\tilde{k}^* \\
\Rightarrow &\tilde{c}^* = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{\alpha}{1-\alpha}}\left[1 - \frac{\alpha\,(\delta + n + g_A)}{\delta + \rho + \theta g_A}\right].
\end{aligned} \tag{6.2}$$

The transition dynamics of the model in $(\tilde{c}, \tilde{k})$ space is characterized by the Jacobian of the system of equations (7.2) evaluated in the neighborhood of the steady-state

$$\begin{aligned}
J &= \begin{pmatrix} \frac{\partial\left(\frac{\dot{\tilde{c}}}{\tilde{c}}\right)}{\partial\tilde{\tilde{c}}} & \frac{\partial\left(\frac{\dot{\tilde{c}}}{\tilde{c}}\right)}{\partial\tilde{\tilde{k}}} \\ \frac{\partial\tilde{\tilde{\tilde{E}}}}{\partial\tilde{c}} & \frac{\partial\tilde{\tilde{\tilde{c}}}}{\partial\tilde{k}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{\theta}f''(\tilde{k}) \\ -1 & f'(\tilde{k}) - (\delta + n + g_A) \end{pmatrix}.
\end{aligned}$$

Notice that $\det J < 0$. Since $\det J = \mu_1 * \mu_2$, where $\mu_{1,2}$ are the eigenvalues of the matrix J, we have that μ_1 and μ_2 have different signs. This means that we have saddle path with one stable arm and one unstable arm. The stable arm corresponds to negative eigenvalue, while the unstable arm corresponds to the positive eigenvalue.

The phase diagram of the system is as follows. [2].

The stable arm is the path converging to the steady state (located on the south-west and the north-east quadrant).

[2] See Appendix 6.7.1 for more details to build the phase diagram

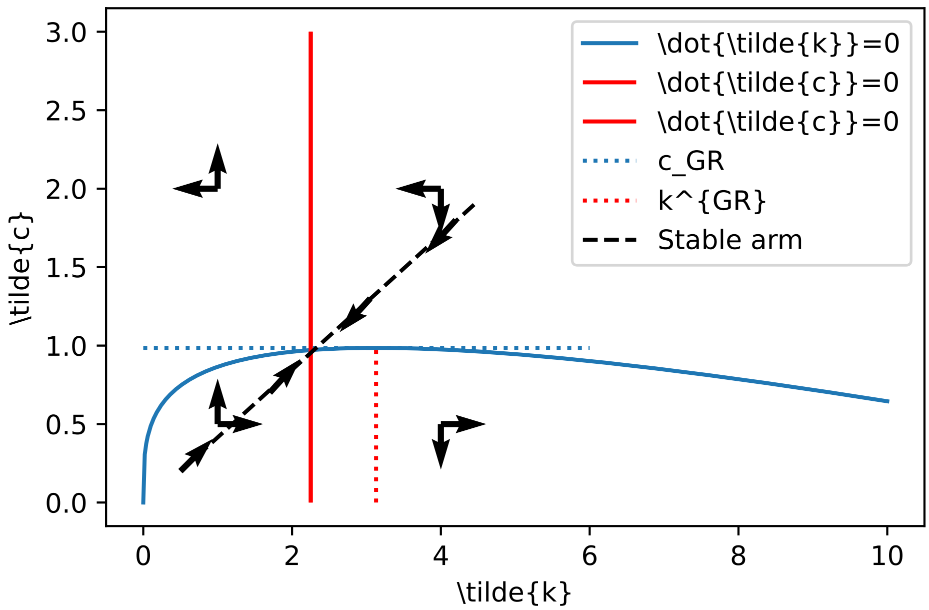


Figure 6.1: Phase Diagram of the Ramsey Model

6.5 The Golden rule

Note that just as in the Solow model, the golden rule is defined by the maximum consumption level in steady state,i.e. the maximum of the curve in Figure 6.1, that is:

$$\tilde{c}^{GR} : \max_{\tilde{k}} \tilde{c} = f(\tilde{k}) - (\delta + n + g_A)\tilde{k} \Rightarrow f'(\tilde{k}^{GR}) = \delta + n + g_A$$

This has to be compared with respect to the steady state in equations (6.2):

$$f'(\tilde{k}^{*}) = \delta + \rho + \theta g_A$$

Hence, remembering that $f'(\tilde{k})$ is decreasing in $\tilde{k}$, the steady state is dynamically efficient (on the left-hand side of the maximum level of k^{GR} in Figure 6.1) if $\delta+\rho+\theta g_A > \delta+n+g_A \Rightarrow \rho > n + (1-\theta)g_A$. Note that given the transversality condition and firms' FOC $r = f'(k) - \delta$, we see that the economy is indeed dynamically efficient as soon as the transversality condition is satisfied (i.e $\rho > n+\theta g_A$). This is in contrast to the Solow model that admits the possibility of a dynamically inefficient economy. The reason of such differences is that households' here behave optimally over *all* generations including therefore not yet born individuals. Note however that due to a positive rate of time preference $\rho > 0$, the golden rule can not be reached (individuals take into account future generations but also favor present consumption). We label therefore the steady state condition for capital as the **modified Golden rule**.

! Important

It is important to precise that although the economy never satisfies the golden rule of capital accumulation, this is not necessarily a problem as this is the optimal behaviour given a welfare criterion, even as new generations arrive continuously and no one dies (see next chapter in the case of finite a lifetime case). In contrast, dynamic efficiency was the only criterion to evaluate an economy in the Solow model but it was not based on a welfare criterion.

6.6 The Social Planner Problem

In the previous section, we were studying the *decentralized* problem, that is when households and firms make choices on their respective side and meet on the different market where prices adjust to ensure equilibrium. In this section, we consider the problem of a planner that take decisions for all agents (without any price mechanisms). We will show that in absence of any distortion in the decentralized case, the two problems are equivalent. The main interest for this exercise is to compare the first-best optimal allocations with respect to a decentralized (potentially distorted) outcome.

A planner have the same preferences as an individual household and firm's technology. It has to choose the path of per capita consumption $c(t)$ and effective capital stock $\tilde{k}(t)$ to maximize:

$$\begin{aligned} \max_{c(t),\tilde{k}(t)} \int_{t=0}^{\infty} u(c) \exp^{-(\rho-n)t}\, dt \\ \text{s.t.} \quad \dot{\tilde{k}}(t) = f(\tilde{k}(t)) - (\delta + n + g_A)\tilde{k}(t) - c(t) exp^{-g_A t} \\ \tilde{k}(0) \text{ given} \end{aligned}$$

where the constraint is the aggregate resource constraint of the economy written in term of effective labor.

Denote λ as the multiplier of the constraint of the current-value Hamiltonian, the FOCs are:

$$\begin{aligned} u'(c(t)) = \lambda(t) \exp -g_A t \\ (\rho - n)\lambda(t) - \dot{\lambda}(t) = \lambda(t)(f'(\tilde{k}(t)) - (\delta + n + g_A)) \\ \Rightarrow -\dot{\lambda}(t) = \lambda(t)(f'(\tilde{k}(t)) - (\rho + \delta + g_A)) \\ \lim_{t\to+\infty} \tilde{k}(t)\lambda(t) \exp -(\rho - n)t = 0 \end{aligned}$$

Log-differentiate the first equation with respect to time to obtain $-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = g_A - \frac{\dot{\lambda}(t)}{\lambda(t)}$ and substitute the growth rate of $\lambda(t)$ by the second Foc:

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = g_A + (f'(\tilde{k}(t)) - (\rho + \delta + g_A))$$

After eliminating the terms in g_A, we note that we obtained the same FOC in the decentralized problem, hence the equivalence (in absence of distortion) of the two problems.

6.7 Appendix

6.7.1 Phase diagram

Consider the dynamic system we obtained:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta}\left[f'(\tilde{k}(t)) - \delta - \rho - \theta g_A\right]$$
$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{f(\tilde{k}(t))}{\tilde{k}(t)} - \frac{\tilde{c}(t)}{\tilde{k}(t)} - (\delta + n + g_A)$$

Consider first the Euler equation. It is easy to show that:

$$\dot{\tilde{c}} > 0 \Leftrightarrow f'(\tilde{k}) > \delta + \rho + \theta g_A \Leftrightarrow \tilde{k} < \tilde{k}^*$$
$$\dot{\tilde{c}} > 0 \Leftrightarrow f'(\tilde{k}) < \delta + \rho + \theta g_A \Leftrightarrow \tilde{k} > \tilde{k}^*$$

Where the last implications are obtained because $f''(\tilde{k}) < 0$. It follows that on the left (respectively right) of the locus defining $\dot{\tilde{c}} = 0$, that is where $\tilde{k} < \tilde{k}^*$ (respectively $\tilde{k} > \tilde{k}^*$), consumption is increasing (respectively decreasing) such that:

Consider now the capital accumulation equation. We get:

$$\dot{\tilde{k}} > 0 \Leftrightarrow \tilde{c} < f(\tilde{k}) - (\delta + n + g_A)\tilde{k} \Leftrightarrow \tilde{k} < \tilde{k}^*$$
$$\dot{\tilde{k}} > 0 \Leftrightarrow \tilde{c} > f(\tilde{k}) - \delta + \rho + g_A)\tilde{k} \Leftrightarrow \tilde{k} > \tilde{k}^*$$

That is for every value of $\tilde{c}$ above (respectively below) the curve, $\tilde{k}(t)$ decreases (respectively increases) such that:

Combining the vectors of each case we just defined gives Figure 6.1.

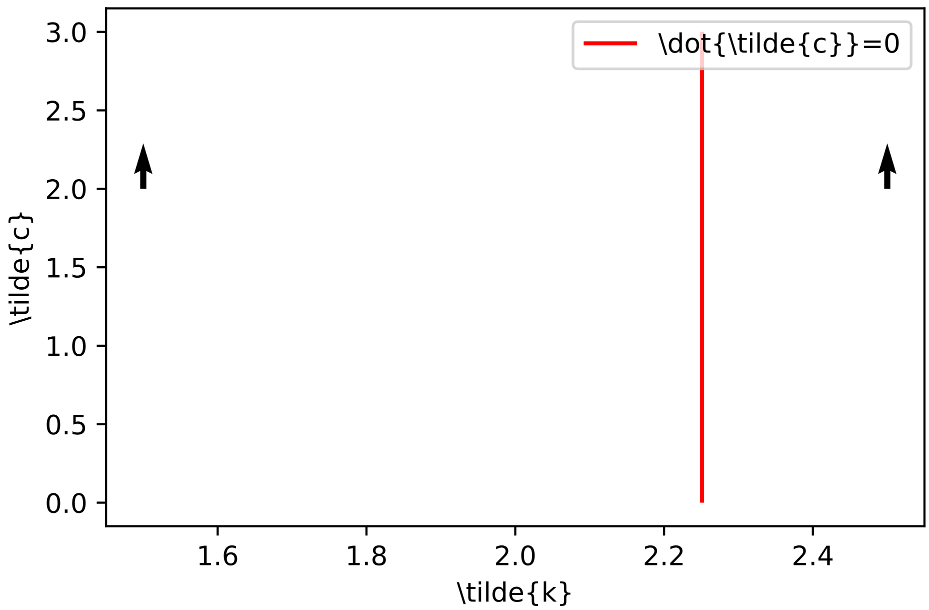


Figure 6.2: Phase Diagram of the Ramsey Model

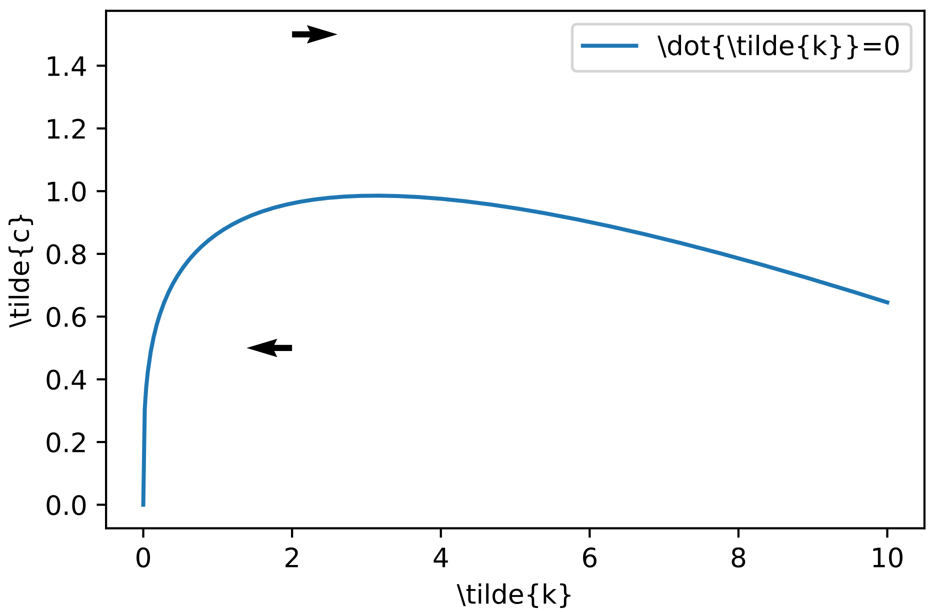


Figure 6.3: Phase Diagram of the Ramsey Model

6.7.2 The discrete-time Ramsey Model

We keep the same assumptions and structure as in the continuous-time model.

The housolds discounted utility function is:

$$\sum_{t=0}^{+\infty} \beta^t u(c_t)$$

with $\beta \in (0,1)$ the discount factor. Population N growing such that $N_t = (1+n)^t N_0$. The household budget constraint per capita and in terms of effective labor are respectively:^{ Note here we assume that households own the capital and lend it to the firms without loss of generality.}

$$\begin{aligned}(1+n)k_{t+1} &= r_t k_t + w_t + (1-\delta)k_t - c_t \\ \rightarrow (1+g)(1+n)\tilde{k}_{t+1} &= r_t \tilde{k}_t + w_t/A_t + (1-\delta)\tilde{k}_t - c_t/A_t\end{aligned}$$

with k_0 given.

The problem of households is to maximize, by choosing the sequence $\{c_t, k_{t+1}\}$, their lifetime discounted utility subject to the per capita budget constraint. The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t \left\{u(c_t) + \lambda_t \left[w_t + r_t k_t + (1-\delta)k_t - c_t - (1+n)k_{t+1}\right]\right\}$$

The FOCs are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_t} &= 0: \quad u'(c_t) = \lambda_t \\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= 0: \quad \lambda_t(1+n) = \beta\lambda_{t+1}(r_{t+1} + (1-\delta)) \\ \lim_{t\to+\infty} k_t \lambda_\beta^t &= 0\end{aligned}$$

Using the first equation into the second, we obtain the discrete-time Euler equation:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{\beta\left(r_{t+1} + (1-\delta)\right)}{1+n} \tag{6.3}$$

If we assumed a CIES formulation for the utility function $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$, we'd get:

$$\left(\frac{c_{t+1}}{c_t}\right)^\theta = \frac{\beta\left(r_{t+1} + (1-\delta)\right)}{1+n} \tag{6.4}$$

Just as in the continuous time case, the firm maximizes its profit by demanding optimal amount of capital K_t and labor L_t. The production technology has labor-augmenting technological progress $A_t = (1+g)^t A_0$.

Hence:

$$\begin{aligned} r_t &= f'(\tilde{k}_t) \\ w_t &= f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \end{aligned} \tag{6.5}$$

In equilibrium, households and firms face the same wage rate and the same interest rate. Using (6.5) into (6.4) and the budget constraint written in terms of effective labor, and transforming consumption per capita in the (6.4) into consumption in terms of effective labor, we obtain the discrete-time dynamic system of the Ramsey model:

$$\begin{aligned} \left(\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right)^\theta &= \frac{\beta\left(f'(\tilde{k}_{t+1}) + (1-\delta)\right)}{(1+n)(1+g)^\theta} \\ (1+g)(1+n)\tilde{k}_{t+1} &= f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - \tilde{c}_t \end{aligned} \tag{6.6}$$

In steady state, $k_{t+1} = k_t = k^*$ and $c_{t+1} = c_t = c^*$ and thus:[3]

$$\begin{aligned} f'(\tilde{k}^*) &= \frac{(1+n)(1+g)^\theta}{\beta} - (1-\delta) \\ \tilde{c}^* &= f(\tilde{k}^*) - (n+g+\delta)k^* \end{aligned}$$

Given $f'(\tilde{k}^*)$ is decreasing, this defines a unique capital stock in the first equation. Given this, the steady consumption is obtained with the second equation. To characterize the BGP, we note that for any variable $x_t \Rightarrow \frac{x_{t+1}}{x_t} = 1 + g_x$. Then, rewrite the Euler equation such that:

$$(1+g_c)^\theta = \frac{\beta\left(f'(\tilde{k}_{t+1}) + (1-\delta)\right)}{(1+n)(1+g)^\theta}$$

As g_c is constant along a BGP, so is $\tilde{k} = \frac{K_t}{A_t L_t}$ and therefore $\frac{Y}{AL} = \tilde{y} = f(\tilde{k})$ is also constant, i.e. $\tilde{y}\tilde{k} = f(\tilde{k})/\tilde{k}$ is constant and therefore $g_k = g_y$. Using the budget constraint:

$$(1+n)(1+g)(1+g_k) = \frac{f(\tilde{k})}{\tilde{k}} + (1-\delta) - \frac{\tilde{c}}{k}$$

[3]We omitted the term ng associated wiht k_{t+1} in the budget constraint

As both g_k and $f(\tilde{k})/\tilde{k}$ are constant, the ratio $\frac{\tilde{c}}{\tilde{k}}$ is also constant implying $g_c = g_k = g_y$. As a result, aggregate variables C, K, Y grows at rate $n+g$ while per capita variable c, k, y grows at rate g.

We study the transitional dynamics by linearizing the system (6.6) in the neighborhood of the steady state. Totally differentiate the system:[4]

$$\theta \frac{d\tilde{c}_{t+1}}{\tilde{c}} - \theta \frac{d\tilde{c}_t}{\tilde{c}} = \frac{f''(\tilde{k}^*)\tilde{k}^* \frac{d\tilde{k}_{t+1}}{\tilde{k}^*}}{f'(\tilde{k}^*) + (1-\delta)} = \frac{\beta f''(\tilde{k}^*)\tilde{k}^* \frac{d\tilde{k}_{t+1}}{\tilde{k}^*}}{(1+n)(1+g)^\theta}$$

$$(1+n)(1+g)\frac{d\tilde{k}_{t+1}}{\tilde{k}^*} = \underbrace{\left(f'(\tilde{k}^*) + (1-\delta)\right)}_{=(1+n)(1+g)^\theta/\beta} \frac{d\tilde{k}_t}{\tilde{k}^*} - \frac{\tilde{c}^*}{\tilde{k}^*}\frac{d\tilde{c}_t}{c^*}$$

Or in matrix form:

$$\begin{pmatrix} \theta & -\frac{\beta f''(\tilde{k}^*)}{(1+n)(1+g)^\theta} \\ 0 & (1+n)(1+g) \end{pmatrix} \begin{pmatrix} \frac{d\tilde{c}_{t+1}}{\tilde{c}^*} \\ \frac{d\tilde{k}_{t+1}}{\tilde{k}^*} \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ -\frac{\tilde{c}^*}{\tilde{k}^*} & (1+n)(1+g)^\theta/\beta \end{pmatrix} \begin{pmatrix} \frac{d\tilde{c}_{t+1}}{\tilde{c}^*} \\ \frac{d\tilde{k}_{t+1}}{\tilde{k}^*} \end{pmatrix} \begin{pmatrix} \frac{d\tilde{c}_{t+1}}{\tilde{c}^*} \\ \frac{d\tilde{k}_{t+1}}{\tilde{k}^*} \end{pmatrix}$$

Inverting the matrix on the left-hand side, we obtain the Jacobian matrix. We can show that the Jacobian matrix has a single eigenvalue within the unit circle, which means that we have a saddle-path.

6.7.3 An alternative resolution: using ratios

In this section, we propose an other approach to characterize the balanced-growth path of the Ramsey model. The strategy relies on rewriting the system in terms of growth rates and ratios that are constantalong the BGP which allows to rewrite the dynamics in a linear form. Assume a Cobb-Douglas technology $Y = K^\alpha (AL)^{(1-\alpha)}$. Typically, the equilibrium path in **per capita** terms can be described by:[5]

$$\frac{\dot{c}(t)}{c(t)} \equiv g_c = \frac{1}{\theta}\left(\alpha \frac{y}{k} - \delta - \rho\right)$$

$$\frac{\dot{k}(t)}{k(t)} \equiv g_k = \frac{y(t)}{k(t)} - \frac{c(t)}{k(t)} - (\delta + n)$$

$$y(t) = k^\alpha A(t)^{1-\alpha} \Rightarrow g_y = \alpha g_k + (1-\alpha) g_A$$

[4] In our case, we are going to log-differentiate as it is easier to write the system in terms of $\frac{dx_{t+1}}{x}$ with $x_t = \tilde{c}_t, \tilde{k}_t$ and $dx_{t+1} = x_{t+1} - x_t$.

[5] We study this approach under a continuous time formulation. Identical steps can be undertaken in a discrete time framework.

and define $z(t) = \frac{y}{k}$ and $x(t) = \frac{c}{k}$. We have:

$$\begin{aligned} g_z = g_y - g_k \Rightarrow g_z &= -(1-\alpha)(z - (n+\delta) - x) \\ g_x = g_c - g_k \Rightarrow g_x &= \frac{1}{\theta}\left(\alpha \frac{\tilde{y}}{\tilde{k}} - \delta - \rho\right) - z + (\delta + n) + x \\ g_y &= \alpha g_k + (1-\alpha) g_A \end{aligned}$$

which is linear in x and z. It is easy to show that for $g_x = g_z = 0$ (i.e. in steady state), we have by definition $g_y = g_k = g_c = g_A$, where the last equality comes from the third equation of the system above. We may then derive steady state values of x and z by solving the two others equations:

$$z = \frac{(\rho + \delta) + \theta g_A}{\alpha}, \quad x = \frac{(\rho + (1-\alpha)\delta) - \alpha n + (\theta - \alpha) g_A}{\alpha}$$

Furthermore, we can also linearize the expressions of g_z and g_x to study transitional dynamics.

Note that an advantage of this method is that solving using ratios means that we are scale-invariant: it does not matter if the system is written in terms of aggregate variable or per capita or in effective labor (as soon as we keep consistency between the set of equations).

7 The Overlapping Generations Model

The Ramsey-Cass-Koopmans model considers a representative household that lives infinite horizons. In many circumstances, however, the assumption of a representative household is not appropriate. One important set of circumstances that may require departure from this assumption is in the analysis of an economy in which new households are born over time. The arrival of new households in the economy is not only a realistic feature, but it also introduces a range of new economic interactions. In particular, decisions made by older generations will affect the prices faced by younger generations. These economic interactions have no counterpart in the neoclassical growth model. They are most succinctly captured in the overlapping generations (OLG) models introduced and studied by Paul Samuelson and later by Peter Diamond. The OLG model considers infinite agents who only live finite periods. In particular, new individuals are continually being born, and old individuals are continually dying.

The OLG model is useful for a number of reasons. First, it captures the potential interaction of different generations of individuals in the marketplace. Second, it provides a tractable alternative to the infinite-horizon representative agent models. Third, some of the key implications are different from those of the neoclassical growth model (e.g. dynamic inefficiency). Finally, the OLG model provides a flexible framework to study the effects of macroeconomic policies such as national debt and social security.

7.1 The Model

In this economy, time is discrete and runs to infinity. Each individual lives two periods. For the generation born in period t, they live for period t and $t+1$. In period t, they are young generation, and become old generation in period $t+1$. As individuals live only two periods, the economy always have two generations in any period. L_t individuals are born in period t. As in Ramsey model, population grows at rate n, i.e.,

$$L_t = (1+n)L_{t-1}$$

Thus, there are L_t young generation and L_{t-1} $(= L_t/(1+n))$ old generation.

7.1.1 Consumers

Each consumer supplies 1 unit of labor at wage rate W_t when he/she is young and divides the labor income between first-period consumption and saving with interest rate $R_t = (1 + r_t)$. In the second period, the individual simply consumes the saving and any interest he/she earns. Let c_{1t} and c_{2t} denote the consumption in period t of young and old individuals. A representative individual born in period t solves

$$\max_{\{c_{1t}, c_{2t+1}\}} \frac{c_{1t}^{1-\theta}}{1-\theta} + \beta \frac{c_{2t+1}^{1-\theta}}{1-\theta}$$

subject to budget constraint

$$\begin{aligned} c_{1t} + s_t &\leq W_t \\ c_{2t+1} &\leq R_{t+1} s_t \end{aligned}$$

The above problem can be written more compactly by substituting the budget constraints as

$$\max_{s_t} \frac{\left(W_t - s_t\right)^{1-\theta}}{1-\theta} + \beta \frac{\left(R_{t+1} s_t\right)^{1-\theta}}{1-\theta}$$

and consumptions are give by

$$\begin{aligned} c_{1t} &= W_t - s_t \\ c_{2t+1} &= R_{t+1} s_t \end{aligned}$$

First order condition for the optimal saving is

$$\left(W_t - s_t\right)^{-\theta} = \beta R_{t+1} \left(R_{t+1} s_t\right)^{-\theta}$$

Thus the optimal saving s_t is given by

$$s_t = s\left(R_{t+1}\right) W_t$$

where $s\left(R_t\right) = \frac{1}{1+\beta^{-\frac{1}{\theta}} R_{t+1}^{1-\frac{1}{\theta}}}$ indicates the saving rate. Note that, for $\theta = 1$ (the utility is logarithm), the saving rate is just a constant $\frac{\beta}{1+\beta}$. Later we will show that in this case the OLG model is equivalent to the Solow model with saving rate $\beta/(1+\beta)$. Moreover, optimal consumptions are given by

$$\begin{aligned} c_{1t} &= [1 - s(R_{t+1})] W_t \\ c_{2t+1} &= R_{t+1} s(R_{t+1}) W_t \end{aligned}$$

7.1.2 Firms

A representative firm hires labor L_t and rents capital K_t to produce final goods according to the production function $Y_t = F(K_t, A_t L_t)$, where the technology A_t is assumed to follow

$$A_t = (1+g)^t A_{t-1}$$

We assume that the capital is fully depreciated. The firm aims to maximize the profit by choosing L_t and K_t. The optimization problem is

$$\max_{\{L_t, K_t\}} F(K_t, A_t L_t) - W_t L_t - R_t K_t$$

The first order conditions w.r.t. $\{L_t, K_t\}$ are given by

$$\begin{aligned} R_t &= F_K(K_t, A_t L_t) \\ W_t &= F_{AL}(K_t, A_t L_t) A_t \end{aligned}$$

We assume the production function is constant return to scale. Let $f(\tilde{k}) = F\left(\frac{K}{AL}, 1\right)$, where $\tilde{k} = \frac{K}{AL}$. The input demands can be expressed as

$$\begin{aligned} R_t &= f'\left(\tilde{k}_t\right) \\ W_t &= \left[f\left(\tilde{k}_t\right) - f'\left(\tilde{k}_t\right)\tilde{k}_t\right] A_t \end{aligned} \tag{7.1}$$

7.1.3 Competitive Equilibrium

In the competitive equilibrium, consumers and firms achieve the individual optimum. Each market clears. In particular, capital market clearing condition implies

$$K_{t+1} = L_t s(R_{t+1}) W_t$$

According to (7.1), the last equation can be rewritten as

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)} s\left(f'\left(\tilde{k}_{t+1}\right)\right)\left[\frac{f\left(\tilde{k}_t\right) - f'\left(\tilde{k}_t\right)\tilde{k}_t}{f\left(\tilde{k}_t\right)}\right] f\left(\tilde{k}_t\right)$$

The above equation fully describes the dynamics of capital stock.

7.2 Dynamics

- Special Case: $\theta = 1$.

Assume that $\theta = 1$ (utility is logarithm) and the production function takes Cobb-Douglas form, i.e., $F(K, AL) = K^\alpha (AL)^{1-\alpha}$. The saving rate in this case is $s\left(f'\left(\tilde{k}_{t+1}\right)\right) = \frac{\beta}{1+\beta}$. Equation (19) can be reduced into

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)} \frac{\beta}{1+\beta}(1-\alpha)\tilde{k}_t^\alpha \tag{7.2}$$

Note that this expression essentially has the same form as the one derived from the Solow model. Hence, when utility function takes logarithm form, the OLG model is degenerated to the Solow model.

7.3 A word on the general Case

Once we relax the assumptions of logarithmic utility and Cobb-Douglas production technology, a wide range of behaviors of the economy are possible, including multiple equilibria. See Michel and De La Croix (2002)

7.4 Dynamic Inefficiency

Even though in the OLG model, competitive equilibrium is achieved, it turns out that the competitive allocation is not necessarily dynamically efficient, just as in the Solow model. To see this, let us discuss the capital stock at the steady state. For simplicity, we still consider the special case where $\theta = 1$ and $f(\tilde{k}) = \tilde{k}^\alpha$.

From (7.2), we can obtain the steady-state capital stock k^* for the competitive equilibrium from

$$f'\left(k^*\right) = \alpha\left(k^*\right)^{\alpha-1} = (1+g)(1+n)\left(\frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}\right). \tag{7.3}$$

Now consider a social planner's problem:

$$\begin{aligned}
&\max_{\{c_{1t},c_{2t}\}} \sum_{t=0} \beta^t \left(L_t \frac{c_{1t}^{1-\theta}}{1-\theta} + L_{t-1}\phi \frac{c_{2t}^{1-\theta}}{1-\theta} \right) \\
=&\max_{\{c_{1t},c_{2t}\}} \sum_{t=0} [\beta(1+n)(1+g)]^t \left(\frac{\tilde{c}_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+n}\phi \frac{\tilde{c}_{2t}^{1-\theta}}{1-\theta} \right)
\end{aligned}$$

where $\phi > 0$ is the weight that social planner puts on the old generation, $\tilde{c}_{1t} = c_{1t}/A_t, \tilde{c}_{2t} = c_{2t}/A_t$. The resource constraint is

$$L_t c_{1t} + L_{t-1} c_{2t} + K_{t+1} = Y_t$$

Detrending both side with $A_t L_t$ gives us

$$\tilde{c}_{1t} + \frac{\tilde{c}_{2t}}{1+n} + (1+n)(1+g)\tilde{k}_{t+1} = f\left(\tilde{k}_t\right)$$

FOCs w.r.t $\left\{\tilde{c}_{1t}, \tilde{c}_{2r}, \tilde{k}_{t+1}\right\}$ are given by

$$\begin{aligned}
\tilde{c}_{1t}^{-\theta} = \phi \tilde{c}_{2t}^{-\theta} &= \lambda_t \\
(1+n)(1+g)\lambda_t &= \beta \lambda_{t+1} f'\left(\tilde{k}_{t+1}\right)
\end{aligned} \tag{7.4}$$

In the steady state, we have

$$f'(k^{gr}) = \frac{(1+n)(1+g)}{\beta}$$

which is the golden rule.

Comparing (7.3) with (7.4), the capital stock in competitive equilibrium is efficient only if

$$\frac{(1+\beta)\alpha}{1-\alpha} = 1$$

Therefore, in general, the competitive equilibrium in the OLG model is not dynamically efficient in contrast to the Ramsey model. This is mainly due to the finite-horizon of households which prevents violating any transversality condition.

Part III

Endogenous Growth Theory

8 The AK model - Spillovers à la Romer(1986)

8.1 Introduction

We started from the standard Solow-Swan growth model. This model, as well as the Ramsey model, has neoclassical production function for the final good. With neoclassical production function these models cannot endogenously generate long run growth since the returns on capital decline with the accumulation of capital because of the decreasing returns assumption.

The model presented below assumes a neoclassical production function - at the "individual level." In addition, it assumes that the labour augmenting technology is a function of average percapita capital stock. While doing so, it has in mind some "learning by doing" effects/spillovers (i.e., the workers learn/become more productive while working with desks, computers, etc.).

8.2 The Model

The main structure of the model is as follows:

- The level of technology/ efficiency that augments the labour input in the production is a function of the average capital-labour ratio in the economy. The motivation for this is that investment of a firm brings productivity gains from its use from the labour. The firm builds up the knowledge (technical expertise) of how to efficiently use the capital by accumulating it, i.e., there is "learning-by-doing" (e.g., production lines). This learning-by-doing effect has an aggregate impact, when any individual firm's technical efficiency is public knowledge, so that all firms can benefit from the technological advance for the use of capital in the production. This gives the link between " A_i " (i.e., some i th firm's efficiency) and the average capital-labour ratio in the economy.
- The level of technology is assumed to be $A \equiv \bar{A}k$, where k is the average of per-capita capital stock and $\bar{A} > 0$ measures the efficiency of use of the capital
- The production function takes the form: $Y_i = F(K_i, AL_i) = L_i F\left(k_i, \bar{A}k\right) \equiv L_i \bar{A}kf\left(\frac{k_i}{\bar{A}k}\right)$. Thus, there are decreasing returns to capital at the firm-level since the firm is so small that it does not take into account its impact on A. However, there are constant returns to capital in symmetric equilibrium since $k_i = k$.

- One-sector model of growth: $\dot{K} = Y - C - \delta K$
- From the consumption-side, the representative household chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^\infty e^{-(\rho-n)t} u(c) dt$, subject to its budget constraint: $\dot{b}(t) = (r-n)b(t) + w - c$, where $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$.
- Population grows at exogenous rate n

For any individual producer the level of efficiency $A(\bar{A}k)$ is taken as given, along with the prices of inputs, when choosing capital and labour to maximize its per period profits π_i, i.e.,

$$\max_{K_i(t), L_i(t)} \pi_i = F\left(K_i(t), A(t)L_i(t)\right) - R(t)K_i(t) - w(t)L_i(t)$$

Therefore the optimal rules are

$$\begin{aligned} \frac{\partial \pi_i}{\partial K_i} &= 0 \\ \Leftrightarrow R &= F_{K_i}\left(K_i, AL_i\right) \\ &= \frac{\partial L_i \bar{A} k f\left(\frac{k_i}{\bar{A}k}\right)}{\partial K_i} \\ &= f'\left(\frac{k_i}{\bar{A}k}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \pi_i}{\partial L_i} &= 0 \\ \Leftrightarrow w &= F_{L_i}\left(K_i, AL_i\right) \\ &= AF_{AL_i}\left(K_i, AL_i\right) \\ &= \frac{\partial L_i \bar{A} k f\left(\frac{k_i}{\bar{A}k}\right)}{\partial L_i} \\ &= \bar{A} k f\left(\frac{k_i}{\bar{A}k}\right) - k_i f'\left(\frac{k_i}{\bar{A}k}\right) \end{aligned}$$

Since all producers are identical, the equilibrium must be symmetric, i.e., $k_i = k$ for $\forall i$. Therefore, from the asset market equilibrium condition ($b(t) = k$ and $r = R(t) - \delta$) it follows that the net rate of returns on assets in the economy is

$$r = f'\left(\frac{1}{\bar{A}}\right) - \delta$$

The standard optimal consumption path that comes from the intertemporal maximization problem of the households is

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r - \rho)$$

As a result, the equilibrium is characterized by two dynamic equations (i.e., the law of motion of capital in per-capita terms and the optimal consumption path)

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right) - \delta - \rho\right]$$
$$\frac{\dot{k}(t)}{k(t)} = \bar{A}f\left(\frac{1}{\bar{A}}\right) - \delta - n - \frac{c(t)}{k(t)}$$

Meanwhile, the standard TVC applies, that the value of the assets (capital) is equal to zero in the end of the planning horizon

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-(\rho-n)t} = 0$$

8.3 Equilibrium and Balanced Growth Path

Given that in equilibrium the marginal product of capital is independent of the level of per capita capital, it is always constant, i.e.,

$$r = f'\left(\frac{1}{\bar{A}}\right) - \delta$$

The constant returns to capital ensure that there can exist long-run growth driven by capital accumulation in this model. Consumption growth is constant in any equilibrium path, i.e.,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right) - \delta - \rho\right].$$

Increase of the per capita consumption over time requires that the externalities in the capital stock (as measured by $\bar{A}$) be large enough, to increase the net marginal product of capital, $f'\left(\frac{1}{\bar{A}}\right) - \delta$, above the time preference rate, i.e.,

$$f'\left(\frac{1}{\bar{A}}\right) - \delta > \rho$$

Given an initial level (choice) of consumption per capita, $c(0)$, the economy is always along a BGP. In other words, there is no transition dynamics in this model and the economy immediately jumps to a balanced growth path. The proof is offered below. From the first dynamic equation (optimal path of consumption) follows that

$$\int \frac{\dot{c}(t)}{c(t)} dt = \int \frac{1}{\theta} \left[f' \left(\frac{1}{\bar{A}} \right) - \delta - \rho \right] dt \Rightarrow$$
$$\int \frac{1}{c(t)} \frac{dc(t)}{dt} dt = \frac{1}{\theta} \left[f' \left(\frac{1}{\bar{A}} \right) - \delta - \rho \right] t + \tilde{c}_0 \Rightarrow$$

where m_0 is some constant,

$$\int \frac{1}{c(t)} dc(t) = \frac{1}{\theta} \left[f' \left(\frac{1}{\bar{A}} \right) - \delta - \rho \right] t + \tilde{c}_0 \Rightarrow$$
$$\ln c(t) = \frac{1}{\theta} \left[f' \left(\frac{1}{\bar{A}} \right) - \delta - \rho \right] t + \tilde{c}_0 \Rightarrow$$
$$c(t) = c(0) e^{\frac{1}{\theta}\left[f'\left(\frac{1}{A}\right) - \delta - \rho \right] t}.$$

with $c(0) = e^{\tilde{c}_0}$. From the law of motion of per-capita capital it follows that

$$\dot{k}(t) = \left(\bar{A} f \left(\frac{1}{\bar{A}} \right) - \delta - n \right) k(t) - c(0) e^{\frac{1}{\theta}\left[f'\left(\frac{1}{A}\right) - \delta - \rho \right] t}$$

The solution to this differential equation is given by the (linear combination) sum of the general solution of the homogenous differential equation and a particular solution of the non-homogenous differential equation. In other words, solve first

$$\dot{\hat{k}}(t) = \left(\bar{A} f \left(\frac{1}{\bar{A}} \right) - \delta - n \right) \hat{k}(t) \Rightarrow \hat{k}(t) = \hat{k}(0) e^{\left(\bar{A} f \left(\frac{1}{A} \right) - \delta - n \right) t}$$

then find a solution to the general equation. A guess for such solution is

$$\tilde{k}(t) = \tilde{k}_1 c(0) e^{\frac{1}{\theta}\left[f'\left(\frac{1}{A}\right) - \delta - \rho \right] t}$$

where $\tilde{k}_1$ is found by plugging the $\hat{k}$ to the law of motion of per-capita capital after differentiation w.r.t to time, i.e.,

$$
\begin{aligned}
&\tilde{k}_1 c(0)\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]t} = \\
\Rightarrow &\left[\bar{A}f\left(\frac{1}{\bar{A}}\right)-\delta-n\right]\tilde{k}_1 c(0)e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]t} - c(0)e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]t} \\
&\tilde{k}_1 = 1/\left\{\left[\bar{A}f\left(\frac{1}{\bar{A}}\right)-\delta-n\right]-\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]\right\}.
\end{aligned}
$$

Thus the solution of the differential equation is

$$
k(t)=\hat{k}(0)e^{\left[\bar{A}f\left(\frac{1}{\bar{A}}\right)-\delta-n\right]t}+\tilde{k}_1 c(0)e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]t}.
$$

Given that $f''<0$ the following inequalities hold

$$
\bar{A}f\left(\frac{1}{\bar{A}}\right)-\delta-n > f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho > 0
$$

From household's optimization problem, in turn, follows that the rate of return on capital accumulation in terms of utility is

$$
\begin{aligned}
-\frac{\dot{\lambda}(t)}{\lambda(t)} &= r(t)-(\rho-n) \\
\lim_{t\to\infty} k(t)\lambda(t)e^{-(\rho-n)t} &= 0.
\end{aligned}
$$

Therefore, the transversality condition requires that $\hat{k}(0)=0$ and

$$
k(t)=\tilde{k}_1 c(0)e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right)-\delta-\rho\right]t}
$$

which means that per-capita capital grows at the same (constant) rate as the consumption.

Another and more intuitive way to show that the growth rate of per-capita capital is always constant and equal to the growth rate of consumption.

Capital per head will grow at a constant rate only if the consumption-to-capital ratio remains constant over time. In order to examine the properties of the BGP, examine the behavior of the $\frac{c}{k}$ ratio, i.e.,

$$
\frac{(c(t)\dot{/}k(t))}{c(t)/k(t)}=\frac{\dot{c}(t)}{c(t)}-\frac{\dot{k}(t)}{k(t)}=\frac{c(t)}{k(t)}+\frac{f'\left(\frac{1}{\bar{A}}\right)-\theta\bar{A}f\left(\frac{1}{\bar{A}}\right)-(1-\theta)\delta-\theta n-\rho}{\theta}
$$

Note that the above dynamic equation is unstable in $\frac{c(t)}{k(t)}$. Moreover, unless $\frac{\dot{c}(t)}{c(t)}=\frac{\dot{k}(t)}{k(t)}$, the growth rate of $k(t)$ either should cease or should increase to infinity. Both cases would violate transversality condition. Therefore, it must be that the BGP is characterized by constant $(\frac{c}{k})^* = \frac{(1-\theta)\delta+\theta\bar{A}f\left(\frac{1}{\bar{A}}\right)-f'\left(\frac{1}{\bar{A}}\right)+\rho-\theta n}{\theta}$. In the event of a structural change, consumption in this model makes a "discrete" shift to ensure that $\frac{c(t)}{k(t)} = (\frac{c}{k})^*$ and the economy is set again on a BGP.

The steady-state growth rate of per capita consumption and capital is

$$g = \frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{1}{\theta}\left[f'\left(\frac{1}{\bar{A}}\right) - \delta - \rho\right]$$

Furthermore, as in the Solow model, the savings rate in this economy is constant but is an endogenous object since:

$$s = \frac{Y(t) - C(t)}{Y(t)} = 1 - \frac{c^*}{k}\frac{1}{\bar{A}f\left(\frac{1}{\bar{A}}\right)} = \frac{f'\left(\frac{1}{\bar{A}}\right) - (1-\theta)\delta - \rho}{\theta\bar{A}f\left(\frac{1}{\bar{A}}\right)}$$

8.4 Comparative Statics

- Increase in $\bar{A}$ increases g and has ambiguous effect on savings rate s
- Higher $\bar{A}$ implies higher growth rate g since it increases the effectiveness of capital per head in increasing the labor productivity
- Increase in θ or ρ decrease both g and s
- The higher θ and ρ imply lower saving rate s since the first one increases the consumption smoothing and the second one induces higher consumption at current period. In turn, lower savings rate implies lower growth g through lower capital accumulation and, thus, learning-by-doing spillovers.

8.5 Social Planner's Problem

Note that the welfare theorems will not work in this model since knowledge externalities are assumed, which are inherent to the accumulated capital stock and are not internalized in competitive equilibrium. Due to these externalities the competitive equilibrium outcome may not be the first best (the socially optimal one).

The Social Planner (SP) internalizes these externalities and therefore he identifies that $k_i(t) = k(t)$, when considering the marginal product of capital in the production. The SP selects the paths of quantities that deliver maximum utility (social welfare). The SP's problem is

$$
\begin{aligned}
&\max_{c} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-(\rho-n)t} dt \\
&\text{s.t.} \\
&\dot{k}(t) = \bar{A}k(t)f\left(\frac{1}{\bar{A}}\right) - c(t) - (n+\delta)k(t) \\
&k(0) > 0 \text{ given}
\end{aligned}
$$

This problem in terms of current value Hamiltonian is

$$
\max_{c} H_{SP} = \frac{c(t)^{1-\theta}-1}{1-\theta} + \lambda(t)\left(\bar{A}k(t)f\left(\frac{1}{\bar{A}}\right) - c(t) - (n+\delta)k(t)\right).
$$

The first order conditions (optimal rules) are

$$
\begin{aligned}
c(t)^{-\theta} &= \lambda(t), \\
\dot{\lambda}(t) &= \lambda(t)(\rho-n) - \lambda(t)\left(\bar{A}f\left(\frac{1}{\bar{A}}\right) - (n+\delta)\right).
\end{aligned}
$$

Therefore,

$$
\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left(\bar{A}f\left(\frac{1}{\bar{A}}\right) - \delta - \rho\right)
$$

The same way as in competitive equilibrium it can be argued that the capital per head grows at the same (constant) rate as the consumption. Thus,

$$
g^{SP} \equiv \frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{1}{\theta}\left[\bar{A}f\left(\frac{1}{\bar{A}}\right) - \delta - \rho\right]
$$

and

$$
\left(\frac{c(t)}{k(t)}\right)^{SP} \equiv \frac{c(t)}{k(t)} = \frac{(1-\theta)\left(\delta - \bar{A}f\left(\frac{1}{\bar{A}}\right)\right) - \theta n + \rho}{\theta}
$$

- The SP achieves higher growth rate because $\frac{f\left(\frac{1}{\bar{A}}\right)/(1/\bar{A})}{f'\left(\frac{1}{\bar{A}}\right)} > 1$, which holds due to the concavity of the production function. - The social marginal product of capital exceeds the private one, because the latter does not account for the efficiency benefits delivered by the overall level of capital stock in the economy. As a result, the SP saves more since $\left(\frac{c}{k}\right)^* = \left(\frac{c}{k}\right)^{SP} + \frac{\bar{A}f\left(\frac{1}{\bar{A}}\right) - f'\left(\frac{1}{\bar{A}}\right)}{\theta}$, where the second term captures the gap between the social and private returns to savings. Thus, the SP achieves higher long-run growth.

- A growth promoting policy would try to eliminate the difference between the $\left(\frac{c}{k}\right)^{*} =$ $\left(\frac{c}{k}\right)${S P}$. Since $\left(\frac{c}{k}\right)^{*} < \left(\frac{c}{k}\right)^{SP}$ this policy would motivate higher savings in competitive equilibrium. Therefore, a policy could be a simple subsidy to the production that equates the private return to capital to the socially optimal one and finances the subsidy through lump sum tax imposed on households .

9 Human capital and Growth (Lucas, 1988)

9.1 Introduction

In contrast to the models presented so far, Lucas (1988) assumes that there are two types of assets endogenously accumulated in the economy, physical capital and human capital. The idea is very simplistic and says that in addition to producing, for instance, more infrastructure we also produce better (or more) educated workers. The better educated workers, then, produce more, while using the same amount of labor. Therefore, the labor productivity increases, and this, together with the capital accumulation, may enable long run growth.

It is worth emphasizing that the biggest difference between Romer (1986) and Lucas (1988) models is that the latter endogenizes the process of labor productivity growth through human capital accumulation, when the former thinks of spillover effects.

If presented in one sector form, the final good production side and the asset accumulation processes of Lucas (1988) model can be written as

$$Y = AK^{\alpha}H^{1-\alpha} = C + I_K + I_H$$

where H is the human capital input, I_K and I_H are the investments for physical capital and human capital accumulation, i.e.,

$$I_K = \dot{K} + \delta_K K$$
$$I_H = \dot{H} + \delta_H H$$

where δ_K is the depreciation rate physical capital and δ_H is the depreciation rate of human capital. Given that we consider an equilibrium where both assets are accumulated, the returns to both assets should be equal. For the current exercise let $\delta_K = \delta_H \equiv \delta$. This would imply that,

$$
\begin{aligned}
\frac{\partial Y}{\partial K} - \delta = \frac{\partial Y}{\partial H} - \delta \Rightarrow \\
\alpha \frac{Y}{K} = (1-\alpha)\frac{Y}{H} \Rightarrow \\
H = \frac{1-\alpha}{\alpha} K \Rightarrow \\
Y = \left[A \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} \right] K
\end{aligned}
$$

Thus, in terms of structure, the ideas behind the Romer (1986) and Lucas (1988) models are quite similar. Both end up having an aggregate production function with non-decreasing returns to scale, and in particular linear in capital. However, Romer (1986) introduces internal spillover while Lucas (1988) considers external effects through human capital accumulation.

This one sector model was a simple representation of Lucas (1988). The model with corresponding assumptions is the following.

9.2 the Model

This is a two-sector model of growth, where the physical capital is still produced with the same technology as the consumption good, but human capital is produced with a different technology. Human capital is the essential input for the production of new human capital. The motivation for this is that the human capital of one generation is an important factor in affecting the formation of human capital of the later generations. If the production of human capital is within a household, that would be the human capital "embodied" in the parents. If its production is through formal education, then that would be the human capital of the teachers with their methodologies (e.g., books). The accumulation (production) of human capital H follows a law of motion:

$$
\dot{H} = BH_H - \delta_H H,
$$

where H_H is the human capital used for its own production. Every unit of human capital produces $B > 0$ new units of human capital. This stock depreciates at a rate $\delta_H > 0$ (e.g., due to "aging" for instance).

> **Note**
>
> There are no diminishing returns to the production of human capital with this type of production function for the human capital. The non-decreasing returns to the production of human capital will be the engine of long-run growth in this model. The increasing

> stock of human capital drives the accumulation of physical capital and the economy grows indefinitely. If, instead, the production of human capital had decreasing returns to its input, this model would have the same predictions as the Solow-Swan model and it would not be able explain growth in the long-run.

The production of final output combines physical capital stock and human capital H_Y, i.e., $Y = AK^{\alpha}H_Y^{1-\alpha}$, where H_Y is the human capital used in production of final good. Standard neoclassical assumptions apply.

The representative household chooses its consumption path, the assets (physical and human capital) in the next period and the allocation of its human capital input between final good and human capital production, in order to maximize its lifetime utility $U = \int_0^{\infty} u(C)e^{-\rho t}dt$, subject to standard budget constraint and the law of motion of human capital, where $u(C) = \frac{C^{1-\theta}-1}{1-\theta}$.

Define the fraction of human capital used in the production of final output as $u \equiv \frac{H_Y}{H}$ and its complement $1-u = \frac{H_H}{H}$. There are no externalities involved in the input and output markets. By the first welfare theorem it is known that the competitive equilibrium will achieve the first-best allocations. The second welfare theorem implies that one can directly solve for the optimal allocations, as there are prices that will support the competitive equilibrium that achieves such intratemporal and intertemporal allocations.

The intertemporal allocation problem has two controls, consumption and allocation of human capital in the two sectors of production that compete for it. There are two state variables, human and physical capital. Physical capital accumulation requires the saving of output (consumption choices), while the human capital accumulation requires investments in terms of real resources that is to say some human capital needs to be driven out of the production of final output to produce future human capital.

Having in mind the welfare theorems, the representative households problem is

$$
\begin{aligned}
&\max_{u,C} U = \int_0^{\infty} \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \\
&\text{s.t.} \\
&\dot{K} = AK^{\alpha}(uH)^{1-\alpha} - \delta_K K - C, \\
&\dot{H} = B(1-u)H - \delta_H H \\
&K(0), H(0) > 0 \text{ given.}
\end{aligned}
$$

Let λ_K and λ_H be the shadow prices for the physical and human capital, respectively. The problem, if written in terms of current value Hamiltonian, is given by

$$\max_{u,C} H_{LC} = \frac{C^{1-\theta}-1}{1-\theta} + q_K\left[AK^{\alpha}(uH)^{1-\alpha} - \delta_K K - C\right] + q_H\left[B(1-u)H - \delta_H H\right]$$

The optimal rules are

$$
\begin{aligned}
C^{-\theta} &= \lambda_K, \\
\lambda_K(1-\alpha)\frac{Y}{u} &= \lambda_H BH, \\
\dot{\lambda}_K &= \rho\lambda_K - \lambda_K\left(\alpha\frac{Y}{K} - \delta_K\right) \\
&= -\lambda_K\left(\alpha\frac{Y}{K} - \delta_K - \rho\right), \\
\dot{\lambda}_H &= \rho\lambda_H - \left\{\lambda_K(1-\alpha)\frac{Y}{H} + \lambda_H\left[B(1-u) - \delta_H\right]\right\}.
\end{aligned}
\tag{9.1}
$$

The standard transversality conditions apply for each of the state variables:

$$
\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\lambda_K(t)K(t) = 0 \\
\lim_{t\to\infty} e^{-\rho t}\lambda_H(t)H(t) = 0
\end{aligned}
$$

Using the first and third equations in (9.1), it follows that

$$
\frac{\dot{C}}{C} = \frac{1}{\theta}\left(\alpha\frac{Y}{K} - \delta_K - \rho\right) \tag{9.2}
$$

While from the second and fourth equations, we get:

$$
\begin{aligned}
\dot{\lambda}_H &= \rho\lambda_H - \left\{\lambda_H\frac{BH}{(1-\alpha)\frac{Y}{u}}(1-\alpha)\frac{Y}{H} + \lambda_H\left[B(1-u) - \delta_H\right]\right\} \\
&= \rho\lambda_H - \left\{\lambda_H Bu + \lambda_H\left[B(1-u) - \delta_H\right]\right\} \Rightarrow \\
\dot{\lambda}_H &= -\lambda_H\left(B - \delta_H - \rho\right)
\end{aligned}
$$

9.3 Equilibrium and Balanced Growth Path

From the optimal consumption path (10.8) and that the growth rate of consumption at steady state should be constant, it follows that the aggregate output Y and capital stock K grow at the same rate, i.e., $g_K = g_Y$. From the resource constraint (or the law of motion of capital) $\frac{\dot{K}}{K} = \frac{AK^{\alpha}(uH)^{1-\alpha}}{K} - \delta_K - \frac{C}{K} = \frac{Y}{K} - \delta_K - \frac{C}{K}$ follows that in steady-state the consumption and capital grow at the same rate, i.e., $g_C = g_K = g_Y$. From the production of human capital, given that B and δ_H are constant parameters and in steady-state $\frac{\dot{H}}{H} =$ is constant, it follows that the share of human capital in production of final good is constant, i.e.,

$$\begin{aligned} \dot{H} &= B(1-u)H - \delta_H H \Rightarrow \\ u &= 1 - \frac{g_H + \delta_H}{B} = \text{ constant} \end{aligned} \tag{9.3}$$

From the production of final good $Y = AK^{\alpha}(uH)^{1-\alpha}$ follows that

$$\frac{Y}{K} = \frac{AK^{\alpha}(uH)^{1-\alpha}}{K} = A\left(u\frac{H}{K}\right)^{1-\alpha}$$

Given that $g_K = g_Y$ and $A, u = \text{const}$ the growth rates of physical and human capital are equal, i.e., $g_K = g_H = g_C = g_Y \equiv g$. From (31) and given that $g_H = g_Y$ and $u, \alpha, B =$ constant follows that

$$\frac{\dot{\lambda}_H}{\lambda_H} = \frac{\dot{\lambda}_K}{\lambda_K}$$

This result should not be a surprising result. Given that both human and physical capital should be accumulated in equilibrium, the rates of return on their accumulation $\frac{\dot{\lambda}_i}{\lambda_i}(i = H, K)$ should be equal. Otherwise, one of these assets will not be accumulated.

This equality implies then that

$$-\frac{\dot{\lambda}_H}{\lambda_H} = (B - \delta_H - \rho) = \left(\alpha\frac{Y}{K} - \delta_K - \rho\right) = -\frac{\dot{\lambda}_K}{\lambda_K}$$

Therefore, from the optimal consumption path (10.8), it follows that

$$g = \frac{1}{\theta}\left(\alpha\frac{Y}{K} - \delta_K - \rho\right) = \frac{1}{\theta}\left(B - \delta_H - \rho\right) \tag{9.4}$$

Thus, given that $g_H = g_C = g$, from (9.3) it follows that

$$\begin{aligned} u^* &= 1 - \frac{(B - \delta_H - \rho) + \theta\delta_H}{\theta B} \\ &= \frac{(\theta - 1)(B - \delta_H) + \rho}{\theta B} \end{aligned}$$

In order to show that $u^* > 0$, consider, for instance, the transversality condition for human capital

$$\lim_{t\to\infty} e^{-\rho t}\lambda_H(t)H(t) = 0$$

Given that in steady-state $\frac{\dot{\lambda}_H}{\lambda_H} = -(B - \delta_H - \rho)$ and $g_H = \frac{1}{\theta}(B - \delta_H - \rho) \Rightarrow$ in order the transversality condition to hold

$$\begin{aligned} -(B - \delta_H - \rho) - \rho + \frac{1}{\theta}(B - \delta_H - \rho) < 0 \Rightarrow \\ \frac{1}{\theta}(B - \delta_H - \rho) < (B - \delta_H) \Rightarrow \\ (\theta - 1)(B - \delta_H) + \rho > 0 \Rightarrow \\ u^* > 0. \end{aligned}$$

Meanwhile, from (9.4) follows that in steady-state

$$\left(\frac{Y}{K}\right)^* = \frac{B - \delta_H + \delta_K}{\alpha}$$

From the law of motion of capital follows that in steady-state

$$\begin{aligned} \left(\frac{C}{K}\right)^* &= \left(\frac{Y}{K}\right)^* - \delta_K - \frac{1}{\theta}\left[\alpha\left(\frac{Y}{K}\right)^* - \delta_K - \rho\right] \\ &= \frac{1}{\theta}\left[(\theta - \alpha)\frac{B - \delta_H + \delta_K}{\alpha} - (\theta - 1)\delta_K + \rho\right] \end{aligned}$$

Therefore the savings rate is

$$s^* = \left(\frac{Y - C}{Y}\right)^* = \frac{\left(\frac{Y}{K}\right)^* - \left(\frac{C}{K}\right)^*}{\left(\frac{Y}{K}\right)^*} = \frac{\alpha(B - \delta_H - \rho) + \alpha\theta\delta_K}{\theta(B - \delta_H + \delta_K)}$$

9.4 Comparative statics

- Increase in B increases g. Ambiguous effects on s^* and u^* (for $\frac{1}{\theta} \geq 1, s^*$ increases and u^* decreases)
- Increase in θ (or ρ) decreases both g and s^*, while it increases u^*
- Increase in α increases s^* but has no effect on g and u^*

9.5 Kaldor stylized facts and first models of endogenous growth

Assume that the aggregate human capital is uniformly distributed across the population: $H = hL$ and there is no population growth. This suggests that the production function may be thought as one with capital and labour. Human capital plays the role of labour-augmenting technological progress that is endogenously generated by savings from the final output production, i.e. $Y = AK^{\alpha}(uhL)^{1-\alpha}$. In equilibrium:

- $g = \frac{\dot{H}}{H} = \frac{\dot{h}}{h}$.
- $Y/L = Au^{1-\alpha}\left(\frac{K}{hL}\right)^{\alpha} h$ increases at a rate g - K/L also increases at a rate g - Y/K is constant
- The real interest rate $r = B - \delta_H$ is constant
- The wage rate $w = \frac{\partial Y}{\partial(uL)} = (1-\alpha)\frac{Y}{H}h$ increases at rate g
- Growth rates across countries differ in the long-run due to technology and preference parameters. Initial conditions (initial levels of human and physical capital stock) have a permanent effect on the level of welfare. When economies start with different endowments, the model predicts no convergence in levels of GDP per capita, even if countries have the same long-run growth rate. Parameter changes explain transition dynamics (not covered here) that can accommodate explanations for short episodes of strong growth.

9.6 Further comments

Lucas motivated the importance of human capital accumulation for long-run economic growth, by forming two different (yet complementary) models. The first model allows for human capital accumulation out of the market (e.g., education sector) that would imply that there is a tradeoff between current consumption and future one, since human capital needs to be driven out of current production sector. Furthermore, in his original specification he allowed for both internal and external returns to human capital in the final-good production (spillovers à la Romer, $Y = (AH^{\gamma})\, K^{\alpha} H_Y^{1-\alpha}; \gamma > 0$).

The second model allows on-the-job accumulation of human capital, i.e., another form of "learning-by-doing". He assumed multiple goods, with different rates of human capital accumulation as byproduct of their production. The trade-off in this case is that human capital accumulation takes a form of a less desirable mix of current consumption goods. Growth promoting policies implied by either model are very different (education subsidies vs. industrial policy).

The research challenge that Lucas acknowledges himself is that human capital is not a measurable factor, and in particular its potential external effects. He proposes that a good example of the importance of external effects of human capital is the formation of cities.

Overall, the model lacks a good justification of the non-decreasing returns to the human capital accumulation. The accumulation of human capital differs importantly from the accumulation of knowledge and therefore this model is not a model of technological progress. The important difference between them is that human capital is rival and excludable while knowledge is not rival, though can be excludable.

Empirically, human capital growth cannot explain cross-country growth differences. There is only some limited support that human capital matters as an input to R&D. The latter comes out an important factor in driving aggregate productivity and explaining the cross-country variation in the growth and levels of GDP per capita.

10 Horizontal Innovations and Model of Expanding Varieties (Romer, 1990)

10.1 Introduction

The next two chapters are devoted to models of endogenous technological change. We have seen in both Solow and Ramsey model that technical progress, left unexplained for now, is the main driver of economic growth once an economy as reached the steady state. Furthermore, as we have seen in the last previous chapters, the AK structure is particularly useful to provide sustained growth. We need therefore to understand how economic decisions lead to sustained growth and thus to open the "black box". Two types of technological change is considered: horizontal and vertical innovations, which can be summarized as follows:

- horizontal innovations explain technological changes as appearing from the creation of new products
- vertical innovations look at technological changes as the result of improving quality of existing good (or decreasing cost of production).

While certainly not orthogonal, we discuss vertical innovations in the next chapter. The present chapter is devoted to horizontal innovations as introduced by Romer (1990).

The main ideas behind Romer (1990) model are:

1. technology is an important factor in production
2. technological progress is market outcome, i.e., it is endogenously generated
3. technology is a good with special features:
    - technology is not rival, i.e., if it is created it can be used at zero cost any time after by everyone
    - technology is at least partly excludable, i.e., one can restrict the access of others to its technology, to some extent (thus s/he can earn returns)

10.2 The Model

This is a multi-sector model of R&D-based endogenous growth

The sectors are:

1. R&D sector that produces "blueprints" of new varieties/types of capital goods $\dot{A}$. The R&D production uses L_A amount of total labour L. The existing set of varieties A increases the productivity of the R&D sector, i.e., positive knowledge externalities operate in the production of new blueprints, creating increasing returns in this sector

$$\dot{A} = BAL_A, \quad B > 0,$$

where B is the efficiency of "blueprint" creation. - An example could be the creation of wireless telephone while using the knowledge of transmitting information via radio waves and voice encoding.

2. Capital variety producing sector that uses the blueprints and produces intermediate capital goods for the final goods production. It is characterized by monopolistic competition. There is free-entry in the market of new blueprints. Entrepreneurs compete for patent that provides them with infinite-horizon property rights on a new blueprint. The acquisition of a patent allows an entrepreneur to employ exclusively the new blueprint and produce a distinct capital good thereafter. The production of capital goods/varieties requires investment in terms of the (foregone) final good. Romer motivates the price-setting assumption by entry (fixed) costs. Each and every capital good producing firm first buys (invests) the blueprint of capital good. It then enters to capital goods market and stays there forever. In capital goods market the firm has to have strictly positive profit streams in order to recover the entry cost. To have positive profits, it has to be a price setter. Moreover, in free entry equilibrium the investment cost equals to the value derived in the market, i.e., the present value of discounted profit streams.

3. Final good production sector where producers employ L_Y amount of total labor L and varieties (defined as a set) of capital goods $x(i), i \in [0, A]$ in the production, that is:

$$Y = L_Y^{1-\alpha} \int_0^A x(i)^\alpha di$$

These firms are fully competitive in input and output markets.

The final good is the numeraire and may be either consumed or invested.

4. On the consumption-side, the representative household chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^\infty u(C)e^{-\rho t}dt$, subject to the standard budget constraint.

We also assume that all capital varieties depreciate fully within one period and there is no growth in population, i.e., L is constant.

The following sections develop and solve the model.

> **Note**
>
> This is the benchmark model. Two others versions show the robustness of the benchmark Romer (1990) model: the "lab-equipment" model where the final output is used to create new varieties and the "labor-for-intermediates" where labor is not embodied in the final good but is the unique input to produce the intermediate good. For sake of brevity, we redirect the reader toward Gancia and Zilibotti (2005).

10.3 Behaviors and Market equilibrium

10.3.1 Final goods production

The final good producers maximize their profits taking the price of their inputs, labour (w) and capital goods/varieties ($p_x(i), \forall i$) as given. The problem of the representative final good producer is

$$\max_{\{x(i)\}_{i\in[0,A]}, L_Y} L_Y^{1-\alpha} \int_0^A x(i)^\alpha di - \int_0^A p_{x(i)} x(i) di - wL_Y$$

The FOC are

$$\begin{aligned} F_L &= \frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} = w, \\ F_{x(i)} &= \frac{\partial Y}{\partial x(i)} = \alpha L_Y^{1-\alpha} x(i)^{\alpha-1} = p_x(i); \forall i, \end{aligned} \tag{10.1}$$

where the first expression describes the demand for labour and the second describes the demand for a capital good.

10.3.2 Individual variety

Each capital variety producer i, within every period maximizes her profits $\pi_x(i)$, by selecting the price $p_x(i)$ and the quantity of production $x(i)$. For every unit of capital that it produces it needs to invest one unit of final good that it "borrows" from households at the current price of final output (which is set to one, i.e., the final good is the numeraire), i.e., $\pi_x(i) =$

$p_x(i)x(i) - x(i)$. The firm takes as given the price of the output it uses in the production and the demand that it's good is facing from the final good producers. Since the firm does not have dynamic constraints it's problem is[1]

$$\max_{p_x(i),x(i)} \{\pi_x(i) = p_x(i)x(i) - x(i)\}$$
$$\text{s.t.}$$
$$p_x(i) = \alpha L_Y^{1-\alpha} x(i)^{\alpha-1}$$

The optimal rule(s) are derived by plugging the inverse demand function of capital good to the profit function and taking the derivative with respect $x(i)$, i.e., solve the following problem

$$\begin{aligned} &\max_{x(i)} \left\{\alpha L_Y^{1-\alpha} x(i)^{\alpha} - x(i)\right\} \\ \Rightarrow 1 = \alpha^2 L_Y^{1-\alpha} x(i)^{\alpha-1} &\Rightarrow x(i) = \alpha^{\frac{2}{1-\alpha}} L_Y \end{aligned} \tag{10.2}$$

From (10.1) and (10.2), we derive:

$$\begin{aligned} p_x(i) &= \alpha L_Y^{1-\alpha} x^{\alpha-1} = \frac{1}{\alpha} \Rightarrow \\ \pi_x(i) &= [p_x - 1]\, x = \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} L_Y > 0 \end{aligned} \tag{10.3}$$

Given the symmetry across the different varieties in the final good production (as (10.2) does not depend on index i), the equilibrium implies that all capital good varieties producers will make the same optimal price and quantities choices, i.e., $p_x(i) = p_x$ and $x(i) = x \forall i$.

Note that the imperfect competition in the market equilibrium implies that the price of capital good is a constant mark-up $\left(\frac{1-\alpha}{\alpha}\right)$ above its marginal cost, and the quantity of supplied capital good will be lower than the one selected in a perfectly competitive market (first-best outcome). Note also that in equilibrium the final output is linear in technology since $Y = Y = L_Y^{1-\alpha} \int_0^A x^\alpha di$ for all i which implies $Y = L_Y^{1-\alpha} A x^\alpha = \alpha^{\frac{2\alpha}{1-\alpha}} L_Y A$, and the economy can experience long-run growth driven by technology progress, that here takes the form of expanding variety of capital goods.

[1] It is worth to note that the objective function is (10.4) and in order to solve the optimal problem we could use Hamiltonian. However, since there are no dynamic constraints such solution is tantamount to the proposed one.

10.3.3 Firm entry into capital goods market

The potential capital good producer in order to establish its firm competes with other potential producers in bidding for a new blueprint, where the blueprint is produced in fully competitive market. It makes an up-front (prior to entry) payment for the blueprint. In free entry equilibrium

this payment (cost of entry) is equal to the value derived by the firm in the capital market,

$$V_x(t) = \int_t^\infty \pi_x(\tau) e^{-\int_t^\tau r(s)ds} d\tau \tag{10.4}$$

where t is the entry date and $r(s)$ is the instantaneous real interest rate that the representative household earns on its asset holdings.

From (10.4), it follows that

$$\begin{aligned}
\dot{V}_x(t) &= -\pi_x(t) e^{-\int_t^t r(s)ds} + \int_t^\infty \pi_x(\tau) \frac{\partial}{\partial t} e^{-\int_t^\tau r(s)ds} d\tau \\
&= -\pi_x(t) + \int_t^\infty \pi_x(\tau) \left\{ \frac{\partial}{\partial t} \left[-\int_t^\tau r(s)ds \right] \frac{\partial}{\partial \left[-\int_t^\tau r(s)ds \right]} e^{-\int_t^\tau r(s)ds} \right\} d\tau \\
&= -\pi_x(t) + r(t) V_x(t)
\end{aligned}$$

which is standard Hamilton-Jacobi-Bellman equation. It can be rewritten as

$$rV_x = \pi_x + \dot{V}_x$$

If $V_x(t)$ is constant in equilibrium over time, then

$$V_x(t) = \frac{\pi_x(t)}{r(t)}$$

This condition implies that at every point in time, the instantaneous excess of revenue over marginal cost must be just sufficient to cover the interest rate cost on the initial investment on a new blueprint. Another way of thinking this is that a household lends V_x to the entrepreneur for him to “buy” a blueprint and establish a firm and then receives in every period the “dividends” that equal to the per period profits.

Under the free-entry the value generated by the entry of a firm $V_x(t)\dot{A}$ is equal to the cost of generating the blueprint wL_A,

$$V_x(t)\dot{A} = wL_A \tag{10.5}$$

10.3.4 The $R\&D$ sector

Any blueprint is owned by a capital good producer which has a value $V_x(t)$. Thus, the price/value of a "blueprint" is $V_x(t)$ and the problem of a "blueprint" producer is

$$\max_{L_A} V_x(t) \underbrace{BAL_A}_{=\dot{A}(t)} - wL_A$$

Assuming fully competitive market with free-entry and zero-profit conditions, we get from (10.5):

$$w(t) = V_x(t)BA(t)$$

10.3.5 Labour market

The labour market equilibrium needs to guarantee that labor is freely mobile between the final good and R&D sectors, i.e. the value of the marginal product of labour is equated in these two sectors. Therefore, the wage rate needs to be equal in both sectors:

$$w(t) = V_x(t)BA(t) = (1-\alpha)\frac{Y(t)}{L_Y} = \alpha^{\frac{2\alpha}{1-\alpha}}(1-\alpha)A(t) \tag{10.6}$$

Thus,

$$V_x(t) = \frac{\alpha^{\frac{2\alpha}{1-\alpha}}(1-\alpha)}{B} \equiv V_x$$

Therefore, indeed $V_x = \frac{\pi_x}{r}$.

From (10.3) and (10.7) it also follows that:

$$\begin{aligned} w = V_x(t)BA(t) &= \frac{\pi_x}{r}BA(t) \\ &= \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}}L_Y\frac{1}{r}BA(t)\frac{Y(t)}{Y(t)} \\ &= \alpha(1-\alpha)\frac{1}{r}BY(t). \end{aligned}$$

Plugging back this expression into (10.7), we get:

$$L_Y = \frac{r}{\alpha B}, \quad L_A = L - L_Y \tag{10.7}$$

10.4 The household side

From the standard household intertemporal maximization problem, it follows that its consumption over time follows the path

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{10.8}$$

The standard transversality condition ensures that the value of the asset holdings of the households is equal to zero in the limit, i.e., the growth of the assets does not exceed the real interest rate.

10.5 Balanced growth path

All variables of the model need to grow at constant rates (BGP).

- For $g_A \equiv \frac{\dot{A}(t)}{A(t)} = BL_A$, to be constant in equilibrium, it must be that the allocation of labour in the research and final good sector are constant over time, i.e., $\dot{L}_Y = 0$. From (10.8), it follows that the interest rate should be constant on BGP and $g_C \equiv \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r - \rho)$. Therefore, from (10.7) it follows that $\dot{L}_Y = 0$.
- From the FOCs of the intermediate capital good producers, it follows that x is time invariant, implying that aggregate capital stock available in the economy at every point in time $K = A(t)x$ grows at rate $g_K = g_A$. The economy grows due to capital-deepening which is entirely driven by the expansion of capital varieties.
- The production function of final goods $Y(t) = \alpha^{\frac{2\alpha}{1-\alpha}} L_Y A(t)$ implies that along the BGP $g_Y = g_A$.
- Households' total assets are $b(t) = V(t)A(t) \equiv V_x A(t) \Rightarrow \dot{b}(t) = V_x \dot{A}(t)$. From households' budget constraint, the law of motion of assets is defined by

$$\dot{b}(t) = r(t)b(t) + w(t)L - C(t)$$

Noting that $rb(t) = rV_x A(t) = \pi_x A(t) = \frac{(1-\alpha)Y(t)}{\alpha}$ and $w(t)L = \frac{(1-\alpha)Y}{L_Y}$, we have:

$$
\begin{aligned}
\dot{b}(t) \equiv V_x\dot{A}(t) =& rV_xA(t) + \frac{(1-\alpha)Y}{L_Y}L - C(t) \\
=& \frac{(1-\alpha)Y(t)}{\alpha} + \frac{(1-\alpha)Y}{L_Y}L - C \\
=& \left(\frac{1}{\alpha} + \frac{L}{L_Y}\right)(1-\alpha)Y(t) - C(t) \\
\Rightarrow g_A =& \left(\frac{1}{\alpha} + \frac{L}{L_Y}\right)(1-\alpha)\frac{Y(t)}{V_xA(t)} - \frac{C(t)}{V_xA(t)}
\end{aligned}
$$

Thus, $g_C = g_A = g_Y = g_K \equiv g$.

Note that in equilibrium, there is a positive relation between growth and the real interest rate, as implied by the Euler equation (10.8), but a negative one implied by the / production-side, as

$$
g_A = B\left(L - L_Y\right) = B\left(L - \frac{r}{\alpha B}\right) \tag{10.9}
$$

The latter is due to the fact that higher interest rate reduces the present discounted value of any capital variety firm. By doing so, it reduces the market incentives to direct (labour) resources away from the final good production into the production of new assets/savings instruments. Along the unique BGP, these two forces are equated implying a unique real interest rate and labour allocation. These two can be derived by equating the (10.8) and (10.9)

$$
\begin{aligned}
\frac{1}{\theta}(r-\rho) = B\left(L - \frac{r}{\alpha B}\right) \Rightarrow \\
r = \frac{\alpha}{\alpha+\theta}(\theta BL + \rho) \\
\Rightarrow L_Y = \frac{\theta BL + \rho}{(\alpha+\theta)B}
\end{aligned}
$$

and therefore

$$
g = \frac{r-\rho}{\theta} = \frac{\alpha BL - \rho}{\alpha + \theta}
$$

The condition for positive long-run growth sets a minimum bound on the scale of the economy: $L > \frac{\rho}{B\theta}$. Note that the transversality condition is always satisfied under this condition, i.e., $r > g_A$.

10.6 Comparative statics

- Increase in B and L increase g, by reducing L_Y
- Increase in θ or ρ decreases g, by increasing L_Y
- Increase in α also increases g, by reducing L_Y

10.7 Kaldor stylized facts and the first models of endogenous growth

- $Y/L = Ax^{\alpha}$ increases at a rate g
- $K/L = A\frac{x}{L}$ also increases at a rate g
- Y/K is constant
- The real interest rate r is constant
- The wage rate $w = (1-\alpha)\frac{Y}{L_Y} = (1-\alpha)\left(\frac{x}{L_Y}\right)^{\alpha} A$ increases at rate g
- Growth rates across countries differ in the long-run due to technology and preference parameters. Long-run growth takes place due to the endogenous expansion of the capital-varieties. Short-episodes of fast growth are due to changes in the underlying parameters and/or the scale of the economy.

The model predicts that there is no convergence in terms of GDP per capita.

10.8 Social Planner's problem

In the decentralized market equilibrium there are two sources of inefficiency that drive the equilibrium growth outcome away from the first-best:

1. Monopoly rights/patents
2. Positive knowledge externalities in the production of new blueprints of capital goods

The social planner faces the following optimal control problem, where the state of the economy is summarized by A

$$\begin{aligned}
&\max_{x(i),L_Y,C} \int_0^\infty \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \\
&\text{s.t.} \\
&L_Y^{1-\alpha} \int_0^A x(i)^\alpha di = C + \int_0^A x(i) di \\
&\dot{A} = B\left(L - L_Y\right) A \\
&A(0) > 0 \text{ given.}
\end{aligned}$$

where the second equation is the resource constraint. The third term in resource constraint is the available capital stock. In this model the capital depreciates in a period; thus, the investment equals to the capital stock.

Let λ_K and λ_A denote the shadow prices of "capital" and "knowledge" respectively. The optimal rules imply the following conditions that govern equilibrium in every point in time

$$\begin{aligned}
&C^{-\theta} = \lambda_K \\
&x(i) = \alpha^{\frac{1}{1-\alpha}} L_Y, \forall i \\
&\lambda_K (1-\alpha) \left(\frac{x}{L_Y}\right)^\alpha = B\lambda_A \\
&\dot{\lambda}_A = \rho\lambda_A - \lambda_K \left[L_Y^{1-\alpha} x^\alpha - x\right] - \lambda_A B\left(L - L_Y\right)
\end{aligned}$$

The first equation shows that the social planner would choose to produce to the point that the marginal product of each capital variety equates its marginal cost (one unit of output), hence $x^{SP} > x$. The second equates the value of the marginal product of labour in the final good and R&D production. It implies that over time, the rates of returns are equated, i.e., $-\frac{\dot{\lambda}_K}{\lambda_K} = -\frac{\dot{\lambda}_A}{\lambda_A}$. From the optimal rule for consumption then follows that

$$\frac{\dot{C}}{C} = -\frac{1}{\theta}\frac{\dot{\lambda}_K}{\lambda_K} = -\frac{1}{\theta}\frac{\dot{\lambda}_A}{\lambda_A}$$

where $\frac{\dot{\lambda}_A}{\lambda_A}$ can be derived from the system above:

$$
\begin{aligned}
\frac{\dot{\lambda}_A}{\lambda_A} &= \rho - \frac{\lambda_K}{\lambda_A}\left[L_Y^{1-\alpha}\left(\alpha^{\frac{1}{1-\alpha}}L_Y\right)^{\alpha} - \alpha^{\frac{1}{1-\alpha}}L_Y\right] - B\left(L - L_Y\right) \\
&= \rho - \frac{\lambda_K}{\lambda_A}\left[L_Y\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}L_Y\right] - B\left(L - L_Y\right) \\
&= \rho - B\left[(1-\alpha)\left(\frac{x}{L_Y}\right)^{\alpha}\right]^{-1} L_Y\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha) - B\left(L - L_Y\right) \\
&= \rho - B\left[\left(\frac{x}{L_Y}\right)^{\alpha}\right]^{-1} L_Y\alpha^{\frac{\alpha}{1-\alpha}} - B\left(L - L_Y\right) \\
&= \rho - B\alpha^{-\frac{\alpha}{1-\alpha}}L_Y\alpha^{\frac{\alpha}{1-\alpha}} - B\left(L - L_Y\right) \\
&= \rho - BL.
\end{aligned}
$$

10.8.1 Balanced growth path

From the blueprint production follows that L_Y is constant on BGP. Given this, the x is also constant. Therefore, from resource constraint follows that the macroeconomic aggregates grow at the same constant rate on balanced growth path. Denote that constant growth rate by g^{SP}.

The socially optimal long-run growth is given by

$$
g^{SP} = \frac{BL - \rho}{\theta}.
$$

Since $\alpha \in (0,1)$ the $g^{SP} > g$,i.e.,

$$
\frac{BL - \rho}{\theta} > \frac{\alpha BL - \rho}{\alpha + \theta}
$$

Moreover, since $g^{SP} > g$ from the "blueprint" production follows that $L_Y^{SP} < L_Y$, i.e., the social planner would allocate more of the labour resources into the production of R&D, since the latter is the engine of growth. Therefore, the growth promoting policies increase the incentives to innovate by subsidies to the production of *R&D* (e.g., subsidies to the employment of labour in R&D) that will make the firms internalize the knowledge externalities they generate by each new variety that they discover. The distortion of the imperfect competition may be alleviated by a subsidy to the purchases of the capital goods and/ or subsidies to the production of final output that would increase the demand of capital goods. These policies would finance those subsidies by lump-sum taxes on household.

11 Vertical innovations and Model of Quality Ladders (Grossman-Helpman ,1991; Aghion and Howitt 1992)

11.1 Introduction

In this chapter, we present another version of technological change modelling, i.e. vertical innovations. Instead of seeing new varieties as the result of innovations (e.g. development of DVD products), vertical innovations explains the increase in quality of existing goods (e.g. from DVD to Blu-Ray). This has led to the so-called Schumpeterian Growth literature as the augmented-quality goods replace existing goods.

11.2 The Model

- A multi-sector model of R&D-based endogenous growth that is driven by "creative destruction."
- There are two factors of production: a fixed amount of labour and fixed number, N, of capital good types. Within each variety j, capital goods differ in their quality.
- Qualities are of distance $q > 1$ of each other. The best quality within every sector j is q^{κ_j}, where $\kappa_j \in \mathbb{N} \cup \{0\}$. The initial quality is normalized to one $\left(\kappa_j\big|_{\kappa=0} = 1\right)$.
- The R&D sector produces "blueprints" for improved quality capital goods of each known variety. The input to the R&D production is investment in units of the final output. For the variety j of quality κ_j, the R&D expenditures are $Z_{j\kappa_j}$. The output of the R&D production is uncertain. The R&D expenditures result in the new variety $\left(\kappa_j + 1\right)$ with probability $p_{j\kappa_j}$. The technology of R&D production is linear in R&D investment,

$$p_{j\kappa_j} = \phi\left(\kappa_j\right) Z_{j\kappa_j} \tag{11.1}$$

As quality improves, new discoveries become more expensive in terms of the required investment of resources, i.e., $\phi'\left(\kappa_j\right) < 0$ gives diminishing returns to R&D input. As the probability depends only on the current quality level, it suggests that innovation occurs like a Poisson process.

> **Note**
>
> Linearity implies absence of congestion. Innovation in each sector is "jumpy" (takes place in a discreet manner), however the existence of many sectors and the Law of Large Numbers ensures a smooth outcome at the aggregate level.

- The discovery of a better quality capital good of a particular variety provides an entrepreneur with monopoly rights over the use of the "blueprint." He produces the distinct capital variety with a linear technology that transforms one unit of final output into one unit of capital good.
- There is free entry into the capital goods industry.
- The final good sector operates under perfect competition. It combines labour, L, with qualityadjusted input $\tilde{X}_j$ of every variety j of the existent (fixed) set of capital varieties, $j \in \{1, \dots, N\}$, i.e.,

$$Y = AL^{1-\alpha} \sum_{j=1}^{N} \tilde{X}_j^{\alpha}$$

There is additive separability in all varieties of capital and all of them are used in the final good production due to standard neoclassical function assumptions (Inada conditions). - Different qualities of capital goods of a variety j are perfect substitutes of each other. Hence, if κ_j is the best quality known, the total input employed in the final good sector of the j th capital good variety is:

$$\tilde{X}_j = \sum_{k=0}^{\kappa_j} q^k x_{jk}$$

- It is assumed here that only the highest quality capital good survives of each variety of capital goods, hence

$$\tilde{X}_j = q^{\kappa_j} x_{j\kappa_j}$$

This "guess" regarding the properties of the equilibrium path is to be verified below, by examining the conditions that support it.

Note: The survival of the best quality only across sectors means that the model features technological obsolescence. This results in that the decision to conduct research in order to invent a better quality capital good is based on two forces. First, the discovery is to be overtaken by another researcher in the future (this decreases incentives for research). Second, the discovery of an improved quality capital good implies that there will be a transfer of the monopoly rent from the previous best-quality discovery owner (this increases the incentives for research). - From the consumption-side, the representative HH chooses its consumption and assets to maximize its intertemporal utility: $\int_0^\infty u(C)e^{-\rho t}dt$ subject to the standard budget constraint.

- All capital goods depreciate fully within one period.
- No population growth.

11.3 Market equilibrium

11.3.1 Final goods production

The final good producers maximize their profits taking the price of their inputs, labour (w) and capital goods $\left(P_j\kappa_j, \forall j\right)$ as given. Standard conditions imply that:

$$F_L = \frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} = w$$
$$F_{X_{j\kappa_j}} = \frac{\partial Y}{\partial x_{j\kappa_j}} = A\alpha L^{1-\alpha} x_{j\kappa_j}^{\alpha-1} q^{\alpha\kappa_j} = P_j\kappa_j; \forall j$$

11.3.2 Capital goods production

Each capital good producer, within every period maximizes its profits, $\pi_{j\kappa_j}$, by selecting its price, $P_j\kappa_j$, and quantity of production, $x_{j\kappa_j}$. For every unit of capital that it produces, it needs to invest one unit of final good that it "borrows" from HH at the current output price, i.e.,

$$\pi_{j\kappa_j} = P_j\kappa_j x_{j\kappa_j} - x_{j\kappa_j}$$

The monopolist takes as given the price of the output it uses in its production and the demand that its good is facing from the final good producers, i.e., its problem is

$$\max_{P_j\kappa_j, X_{j\kappa_j}} \pi_{j\kappa_j}$$
$$\text{s.t.}$$
$$P_j\kappa_j = A\alpha L^{1-\alpha} x_{j\kappa_j}^{\alpha-1} q^{\alpha\kappa_j}$$

The FOCs of this optimal program imply

$$x_{j\kappa_j} = LA^{\frac{1}{1-\alpha}} Q^{\frac{2}{1-\alpha}} q^{\frac{\alpha}{1-\alpha}\kappa_j}$$
$$P_{j\kappa_j} = \frac{1}{\alpha}$$

:::{.callout-note} It was assumed that only the best available quality is available within each type of capital good. Suppose instead that this was not the case and the second-best quality is also available. The marginal product of any two consecutive qualities differs by their quality difference, i.e., by a factor of q, which means that the price differential supported by the market equilibrium between the first and second highest quality is: $\frac{P_{j\kappa_j}}{P_{j\kappa_j-1}} = q$. Therefore, the monopoly producer of the second highest quality can at most charge $P_{j\kappa_j-1} = \frac{1}{\alpha q}$. If $\alpha q > 1$, then the second best quality producer cannot cover with such price its marginal cost of production and is driven out of the market. Alternatively, if $\alpha q < 1$, the result that only the leading technology survives within each variety can still be an equilibrium outcome, where the leader follows limit pricing. In such case it charges $P_j\kappa_j = q - \varepsilon$ ($\varepsilon \to +0$) and the next quality producer then could charge $1 - \frac{\varepsilon}{q} < 1$. :::

With the leading technology only, then the final output production in equilibrium is described by

$$Y = AL^{1-\alpha} \sum_{j=1}^{N} q^{\alpha\kappa_j} x_{j\kappa_j}^{\alpha}$$
$$= LA^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} \sum_{j=1}^{N} q^{\frac{\alpha}{1-\alpha}\kappa_j}$$

11.3.3 the $R\&D$ Sector

Given the equilibrium outcome of the capital goods, labour and final output market, the next step is to examine the decision of the entrepreneurs to conduct R&D for the discovery of the κ_{j+1} quality of every type of capital good. Access to the market is free, therefore, every entrepreneur should be in the limit equating his cost in investing in R&D with his expected profit.

The cost of R&D is the investment in terms of current output, $Z_{j\kappa_j}$. The probability of a successful innovation of the quality $\kappa_j + 1$ is $p_{j\kappa_j}$ while the expected value from it is $V_{j\kappa_j+1}$. More rigorously, $V_{j\kappa_j+1}$ is the expected present value of the profit flows of the producer of $\kappa_j + 1$, which within every period is $\pi_{j\kappa_j+1} = \frac{1-\alpha}{\alpha} x_{j\kappa_j+1}$, until its position is overtaken by the discovery of the next quality of the same type of capital good. The latter depends on the probability of the discovery of the next higher quality, $p_{j\kappa_j+1}$. Therefore, the successful innovator of the $\kappa_j + 1$ quality has $V_{j\kappa_j+1}$ that satisfies in equilibrium:

$$
\begin{aligned}
rV_{j\kappa_j+1} &= \pi_{j\kappa_j+1} - p_{j\kappa_j+1}V_{j\kappa_j+1} \\
\implies V_{j\kappa_j+1} &= \frac{\pi_{j\kappa_j+1}}{r + p_{j\kappa_{j+1}}}
\end{aligned}
\tag{11.2}
$$

The last condition is essentially an arbitrage condition. The entrepreneur should be indifferent between lending $V_{j\kappa_j+1}$ units of output and earning the market interest rate, i.e. $rV_{j\kappa_j+1}$ and holding the firm that provides him with a profit flow, $\pi_{j\kappa_j+1}$, and there is a probability that at the end of the period he loses the value of its firm because it is overtaken by a new higher quality product (i.e., its capital loss will be $-V_{j\kappa_j+1}$).

Therefore, given that there is free entry into capital goods sector in equilibrium

$$
p_{j\kappa_j}V_{j\kappa_j+1} = Z_{j\kappa_j}
$$

From this condition and the *R&D* production function (11.1) together with (11.2) it follows that

$$
p_{j\kappa_j+1} = \phi\left(\kappa_j\right)\pi_{j\kappa_j+1} - r
$$

Therefore, the equilibrium probability of a new innovation is influenced by two forces

1. A higher quality of a type of capital variety implies higher demand for this type of capital and thereby higher profits for the successful innovator.
2. The marginal cost of discovering a higher quality capital variety increases.

When the positive effect dominates, newer sectors grow faster than older ones, i.e. there are increasing returns to scale. When the negative effect dominates there is convergence to à la Ramsey. When the two effects balance each other out (i.e. when there are CRS), there is balanced growth in all sectors.

11.4 Balanced growth path

All variables of the model need to grow at constant rates (BGP). A specification for $\phi\left(\kappa_j\right)$ that ensures balanced growth is the following

$$
\phi\left(\kappa_j\right) = \frac{1}{\zeta}q^{-(\kappa_j+1)\frac{\alpha}{1-\alpha}}
$$

This implies that the free-entry condition boils down to

$$\frac{1}{\zeta}\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} = r + p_{j\kappa_j+1} = r + p$$

Note that this specification was chosen in order to eliminate the asymmetry across the different sectors of the economy, given that it is sufficient for a constant probability of new innovations taking place across all sectors. The equilibrium R&D investment is

$$Z_{j\kappa_j} = \frac{p}{\frac{1}{\zeta}q^{-(\kappa_j+1)\frac{\alpha}{1-\alpha}}} = q^{(\kappa_j+1)\frac{\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - r\zeta\right)$$

Note that there are scale effects in this model, as larger sectors spend more on R&D.

Define the aggregate quality index as

$$Q \equiv \sum_{j=1}^{N} q^{\kappa_j\frac{\alpha}{1-\alpha}}$$

Then, the aggregate output and quantity of capital goods are given by

$$Y = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}Q$$
$$X = \sum_{j=1}^{N} X_{j\kappa_j} = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}Q$$

Aggregate R&D expenditures are

$$Z = \sum_{j=1}^{N} Z_{j\kappa_j} = \left(\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - r\zeta\right)q^{\frac{\alpha}{1-\alpha}}Q$$

Aggregate output, capital and R&D expenditures are proportional to the quality index Q, which is itself function of time. Therefore, in steady-state

$$g_Y = g_Z = g_X = g_Q \equiv g$$

The resource constraint of this economy is

$$Y = C + X + Z$$

implying that in the steady-state it is also true that

$$g_C = g$$

On average in the economy in all different capital goods "industries", at every point in time, an innovation of a higher quality capital good takes place with probability p. Therefore, for a large N the Law of Large Numbers provides with the growth rate

$$g^* \approx E\left(\frac{\Delta Q}{Q}\right) = \frac{\sum_{j=1}^{N} p\left(q^{(\kappa_j+1)\frac{\alpha}{1-\alpha}} - q^{\kappa_j\frac{\alpha}{1-\alpha}}\right)}{\sum_{j=1}^{N} q^{\kappa_j\frac{\alpha}{1-\alpha}}} = p\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)$$

This growth rate is implied from the production-side of the economy. It is negatively related to r since $p = \frac{1}{\zeta}\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - r$. On the other hand, the standard optimization condition for the representative household gives a positive relation between the growth rate of the economy and the real return on assets, $g = \frac{1}{\theta}(r-\rho)$. Therefore, the equilibrium interest rate and growth rate are

$$\begin{aligned} r &= \theta p\left(q^{\frac{\alpha}{1-\alpha}} - 1\right) + \rho \\ &= \frac{\theta\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)\frac{1}{\zeta}\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} + \rho}{1 + \theta\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)} \\ g &= \frac{1}{\theta}\left(r^* - \rho\right) \end{aligned}$$

TVC is satisfied for parameters that ensure $r > g$.

11.5 Further comments

The decentralized equilibrium in this model does not achieve first best allocations due to the following distortions

1. Monopoly pricing in the capital goods sector,
2. Obsolescence of the lower quality capital goods.

The first distortion implies that there is low investment and growth in the economy. The second implies that when a new innovation is made, then the successful innovator takes over the profits made by the previous leader in the particular type of capital variety. This "rat race" effect boosts the incentive to conduct $R\&D$ for the purpose of product innovation. At the same time though, for the same potential innovator there is positive probability that himself will be overtaken by the next discovery, which reduces the incentives for R&D. Because of discounting, as current profits matter more than future ones, the outcome is that in equilibrium there is more than optimal $R\&D$ investment, which would tend to increase growth. The net effect of the two distortions is ambiguous.

The decentralized growth rate will be equal to the social planner's one when relative prices are corrected and the successful innovators compensate their immediate predecessors for the loss of their monopoly rents. In the case that all innovations are conducted by the leaders in each sector, i.e. when there is no need for compensation, equilibrium $R\&D$ would be too low. If instead relative prices are corrected, but no compensation is offered, then $R\&D$ would be too high.

Contrast with Romer (1990) The horizontal expansion in the capital good varieties model may be better suited for large-scale inventions (e.g. the ones implying the establishment of a new industry). The vertical expansion of every variety of capital goods is better suited to account for the smaller and gradual improvements in the quality of the capital goods.

On the one hand, both of these endogenous growth models share key features and thereby predictions. In both models, the production-side equilibrium implies that the interest rate has a negative growth effect as it reduces the present discounted value of the expected profits from innovations. Also, both models "suffer" from scale-effects. The engine of long-run growth is the R&D production that is conducted given market-based incentives. Noteworthy, the organizational capital/institutional level of the economy, as captured by the term A in the aggregate production function, has not only a level effect (as in Solow), but a permanent growth effect.

On the other hand, there are important differences in their specifications. Romer's model lacks the feature of creative destruction as it assumes that the different varieties are not direct substitutes or complements. As a result, in the decentralized equilibrium the R&D effort cannot be too high.